One-dimensional solute dispersion along unsteady flow through a heterogeneous medium, dispersion being proportional to the square of velocity

Atul Kumar \(^{a}\), Dilip Kumar Jaiswal \(^{a}\) & Naveen Kumar \(^{b}\)

\(^{a}\) Department of Mathematics & Astronomy, Lucknow University, Lucknow, 226007, India

\(^{b}\) Department of Mathematics, Faculty of Science, Banaras Hindu University, Varanasi, 221005, India

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One-dimensional solute dispersion along unsteady flow through a heterogeneous medium, dispersion being proportional to the square of velocity

Atul Kumar1, Dilip Kumar Jaiswal1 and Naveen Kumar2

1Department of Mathematics & Astronomy, Lucknow University, Lucknow-226007, India
atul.tusaar@gmail.com
2Department of Mathematics, Faculty of Science, Banaras Hindu University, Varanasi-221005, India

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Abstract One-dimensional solute transport, originating from a continuous uniform point source, is studied along unsteady longitudinal flow through a heterogeneous medium of semi-infinite extent. Velocity is considered as directly proportional to the linear spatially-dependent function that defines the heterogeneity. It is also assumed temporally dependent. It is expressed in both the independent variables in degenerate form. The dispersion parameter is considered to be proportional to square of the velocity. Certain new independent variables are introduced through separate transformations to reduce the variable coefficients of the advection–diffusion equation to constant coefficients. The Laplace Transformation Technique (LTT) is used to obtain the desired solution. The effects of heterogeneity and unsteadiness on the solute transport are investigated.

Key words advection; diffusion; heterogeneous medium; degenerate form

Dispersion de soluté à une dimension dans un écoulement non permanent à travers un milieu hétérogène, la dispersion étant proportionnelle au carré de la vitesse

Résumé Nous avons étudié le transport à une dimension de soluté, provenant d’une source ponctuelle uniforme et continue, le long d’un écoulement longitudinal non-permanent dans un milieu hétérogène semi-infini. La vitesse est considérée comme directement proportionnelle à la fonction spatiale linéaire définissant l’hétérogénéité et que l’on suppose aussi dépendre du temps. Elle s’exprime selon les deux variables indépendantes dans la forme dégénérée. Le paramètre de dispersion est considéré comme étant proportionnel au carré de la vitesse. Certaines nouvelles variables indépendantes sont introduites par des transformations distinctes afin de réduire les coefficients variables de l’équation d’advection-diffusion à des constantes. La transformation de Laplace (LTT) est utilisée pour obtenir la solution désirée. Les effets de l’hétérogénéité et de l’instabilité sur le transport de soluté ont été étudiés.

Mots clefs advection; diffusion; milieu hétérogène; forme dégénérée

1 INTRODUCTION

The initial solute transport studies established (Ebach and White 1958) that the longitudinal dispersion coefficient is linearly proportional to the product of the fluid velocity and the particle diameter. This proportionality extends over a broad range of Reynolds numbers. The longitudinal coefficient of dispersion for one-dimensional flow was experimentally determined by Rumer (1962) within a certain range of Reynolds numbers. In agreement with previous investigators, the dispersion coefficient was correlated with the seepage velocity according to the relationship \( D = \alpha w_s^n \), in which \( w_s \) is the average seepage velocity and \( \alpha \) and \( n \) are constants dependent primarily on the geometry of the pore system. In addition, it was found that the preceding equation established for steady flow was also valid for an unsteady flow with
exponential and sinusoidally varying seepage velocities. For \( n = 1 \), this establishes a direct relationship between the dispersion coefficient and the velocity. Taking advantage of this relationship, it became possible to obtain analytical solutions to a class of dispersion problems with unsteady flow. This was accomplished by introducing a new time scale.

In addition to the above theory, two more dispersion theories, and the situations in which they are valid, are widely reported in the literature. According to Scheidegger (1957), the dispersion parameter varies with the square of the velocity. Later, Matheron and deMarsily (1980) observed that in large subsurface formations, dispersivity may vary with position or time along uniform flow.

It is much easier to discover and establish dispersion theories in surface water and atmospheric media than in subsurface formations and aquifers. We have recently reviewed important hydro-dynamic dispersion problems based on these theories and tackled them analytically and numerically (Jaiswal et al. 2009, Kumar et al. 2009, 2010). Analytical solution of a physical problem is of fundamental importance in explicitly understanding the role of all the parameters in the physical phenomenon. The nature of the medium, whether it is porous or non-porous, its heterogeneity, the source type of the pollution, nature of the flow field transporting the solute particles, nature of the pollutants, and the dimension of the problem, are key elements of any hydro-dynamic dispersion problem. Real situations make the parameters of these elements more complex and obtaining an analytical solution difficult, even in one-dimensional space. Remedies in the form of numerical methods (now possible due to the advent of fast computing) are frequently adopted by researchers, but such solutions require analytical solution of a similar less general problem for validation in terms of convergence and stability.

Some recent dispersion problems dealing with heterogeneous media are reviewed here. Liu et al. (2000) applied the generalized integral transform technique to solve the one-dimensional advection–dispersion equation in heterogeneous porous media coupled with either linear or nonlinear sorption or decay, and with spatially and temporally variable flow and dispersion coefficient, and arbitrary initial and boundary conditions. Sander and Braddock (2005) presented a range of analytical solutions for combined transient water and solute transport for horizontal flow, adopting the concept of a scale and time-dependent dispersivity for contaminant transport in aquifers. They applied these solutions to the transient, unsaturated horizontal flow to develop similarity solutions for both constant solute concentration and solute flux boundary conditions. Lowry and Shu-Guang Li (2005) presented a method for solving the time-dependent advection–diffusion equation that does not discretize the derivative terms, but rather solves the equation analytically in the space–time domain. Moreira et al. (2006) proposed a hybrid technique for the unsteady two-dimensional advection–diffusion equation utilizing the GILTT (Generalized Integral Laplace Transform Technique). Moreira et al. (2009) presented a review of the GILTT solutions focusing the applications on pollutant dispersion in atmosphere. Guerrero et al. (2009) presented an analytical methodology by using change-of-variables in combination with GILTT, to solve the advection–diffusion equation in a finite domain for both transient and steady-state regimes. Cassol et al. (2009) presented an analytical solution for the two-dimensional atmospheric pollutant dispersion problem utilizing GITT, the Laplace transform and matrix diagonalization. Kartha and Srivastava (2008) studied the effect of immobile water content on contaminant advection and dispersion in unsaturated porous media. Kuntz and Grathwohl (2009) investigated the transport and fate of reactive components in unsaturated subsoil (the vadose zone) using numerical simulation of steady-state and transient flow scenarios.

In the present work, an analytical solution is obtained for a one-dimensional advective–diffusive equation (ADE) describing the solute transport in a realistic scenario, by using a Laplace Integral Transformation Technique (LITT). This technique is simpler and more viable than those used in previous dispersion models. The two coefficients of the ADE are considered as functions of both the independent variables (space variable and time variable). The spatial dependence is due to the heterogeneity of the medium through which the transport takes place. The temporal dependence is due to unsteadiness of the flow field. The dispersion parameter is considered to be proportional to the square of the velocity (Scheidegger 1957). To use the LITT, the ADE with the variable coefficients is reduced to a form with constant coefficients in terms of new independent variables. The present solution may be used more effectively than previous ones to construct the mass transport function for a new type of transient infinite element (Zhao and Valliapan 1994, Zhao 2009), and other numerical solutions in a semi-infinite domain.
2 MATHEMATICAL FORMULATION AND ANALYTICAL SOLUTION

The linear advection–diffusion partial differential equation in one dimension in general form may be written as:

\[
\frac{\partial c}{\partial t} = \frac{\partial}{\partial x} \left( D(x,t) \frac{\partial c}{\partial x} - u(x,t)c \right) \tag{1}
\]

where \( c \) is the solute concentration at position \( x \) at time \( t \), \( D(x,t) \) is the solute dispersion parameter and \( u(x,t) \) is the velocity of the medium transporting the solute particles. This equation is solved for a dispersion problem in which both the coefficients remain functions of independent variables.

The medium is considered to be of semi-infinite extent and heterogeneous. In a heterogeneous porous medium, porosity changes with position. As seepage velocity depends upon porosity, so it is non-uniform. In a non-porous medium, such as air or a surface water body, velocity is rarely uniform. As a consequence of the heterogeneity of the medium, the velocity of the flow field transporting the solute particles “down stream” is considered spatially dependent. Its expression of increasing nature is linearly interpolated in the position variable in a finite longitudinal region \( 0 \leq x \leq \ell \) in which concentration values are evaluated (Kumar et al. 2010). Let velocity at the origin \( x = 0 \) of the domain be \( u_0 \), which increases to \( u_0(1 + b) \) at \( x = \ell \), where a real constant \( b < 1 \) ensures that the change in velocity is of small order, i.e. the laminar condition of the flow is not affected. Thus the expression for velocity at any position \( x \), may be linearly interpolated as:

\[
u(x) = \frac{x - \ell}{0 - \ell} u_0 + \frac{x - 0}{\ell - 0} u_0(1 + b)
= u_0(1 + ax)
\tag{2}
\]

where \( a = b/\ell \) is a constant less than 1.0 and serves as a parameter of heterogeneity of the medium. Its different values represent media of varying heterogeneity. Further, the velocity is also considered temporally dependent because it is seldom steady. The expression for velocity is written in degenerate form as:

\[
u(x,t) = u_0 f(mt)(1 + ax)
\tag{3a}
\]

where \( m \) may be termed as an unsteady parameter of dimension inverse of the dimension of \( t \). When choosing an expression for \( f(mt) \), it is ensured that \( f(mt) = 1 \) for \( m = 0 \) and \( t = 0 \). The former case represents the steady flow. The latter case represents the velocity at the initial stage. The solute dispersion parameter is considered proportional to the square of the velocity (Scheidegger 1957), that is we consider:

\[
D(x,t) = D_0 f^2(mt)(1 + ax)^2
\tag{3b}
\]

As \( ax \) and \( mt \) are non-dimensional terms, so the constants \( u_0 \) and \( D_0 \) in equations (3a) and (3b) may be referred to as uniform velocity of dimension \( L T^{-1} \) and the initial dispersion coefficient of dimension \( L^{2} T^{-1} \), respectively. Thus ADE (1) is:

\[
\frac{\partial c}{\partial t} = \frac{\partial}{\partial x} \left(D_0 f^2(mt)(1 + ax)^2 \frac{\partial c}{\partial x} - u_0 f(mt)(1 + ax)c\right)
\tag{4}
\]

To solve equation (4), one initial condition and two boundary conditions are required. It is assumed that initially the medium is solute free. The source of the solute mass is a uniform continuous point source at the origin. It serves as the first boundary condition known as the input condition. A flux type homogeneous condition is assumed at the far end of the medium. Thus the initial condition is:

\[
c(x, t) = 0 \quad (t = 0, x \geq 0)
\tag{5}
\]

the input condition is:

\[
c(x, t) = C_0 \quad (x = 0, t > 0)
\tag{6}
\]

and the second boundary condition is:

\[
\frac{\partial c}{\partial x} = 0 \quad (x \to \infty, t \geq 0)
\tag{7}
\]

Using transformation (Crank 1975):

\[
T^* = \int_0^t f(mt)dt
\tag{8}
\]

The advection–diffusion equation (4) becomes:

\[
\frac{\partial c}{\partial T^*} = \frac{\partial}{\partial x} \left(D_0 f(mt)(1 + ax)^2 \frac{\partial c}{\partial x} - u_0(1 + ax)c\right)
\tag{9}
\]
For an expression of \(f(mt)\), \(m\) will occur in the denominator of \(T^*\), so its dimension will be that of \(t\); hence it is referred to as a new time variable. Also an expression of \(f(mt)\) is chosen such that for \(t = 0\), we get \(T^* = 0\), so that the nature of initial condition does not change. The initial and boundary conditions (equations (5)–(7)) become:

\[
c(x, T^*) = 0 \quad (T^* = 0, x \geq 0) \tag{10}
\]

\[
c(x, T^*) = C_0 \quad (x = 0, T^* > 0) \tag{11}
\]

and

\[
\frac{\partial c}{\partial x} = 0 \quad (x \to \infty, T^* \geq 0) \tag{12}
\]

Then another space variable is introduced through a transformation (Kumar et al. 2010):

\[
X = \frac{1}{a} \log(1 + ax)
\]

Equation (9) becomes:

\[
\frac{\partial c}{\partial T^*} = D_0 f(mt) \frac{\partial^2 c}{\partial X^2} - u_0 f_1(mt) \frac{\partial c}{\partial X} - au_0 c \tag{14}
\]

where:

\[
f_1(mt) = 1 - (aD_0/u_0)f(mt) = 1 - \chi f(mt) \tag{15}
\]

is defined as another time-dependent expression in the non-dimensional term, \(mt\), and \(\chi = (aD_0/u_0)\) is a non-dimensional parameter. Further, the first-order decay term in equation (14) is eliminated by using a transformation:

\[
c = C \exp(-au_0 T^*) \tag{16}
\]

Lastly, with the help of other independent variables introduced through the transformations:

\[
Z = X f_1(mt) \tag{17}
\]

and

\[
T = \int_0^t f_1^2(mt)dt \tag{18}
\]

respectively, the variable coefficients of the advection–diffusion equation are reduced to constant coefficients. Thus the dispersion problem defined by equations (4)–(7) is reduced to an initial and boundary value problem in the \((Z,T)\) domain:

\[
\frac{\partial C}{\partial T} = D_0 \frac{\partial^2 C}{\partial Z^2} - u_0 \frac{\partial C}{\partial Z} \quad (0 \leq Z < \infty, T > 0) \tag{19}
\]

\[
C(Z, T) = 0 \quad (T = 0, Z \geq 0) \tag{20}
\]

\[
C(Z, T) = C_0 \exp(au_0 \gamma T) \quad (Z = 0, T > 0) \tag{21}
\]

\[
\frac{\partial C}{\partial Z} = 0 \quad (Z \to \infty, T \geq 0) \tag{22}
\]

where \(\gamma = (1 - \chi)^{-2}\) in equation (21) is another non-dimensional parameter. In fact, due to the transformations defined by equations (13), (16) and (17), input condition (11) becomes:

\[
C(Z, T^*) = C_0 \exp(au_0 T^*) \tag{23}
\]

where the new time variable \(T^*\) is defined by equation (8). To write it in terms of another time variable \(T\) defined by equation (18), we chose an expression of \(f(mt)\). We consider a decelerating exponential function as:

\[
f(mt) = \exp(-mt) \tag{24}
\]

For this, the two new time variables have the respective expressions:

\[
T^* = \int_0^t \exp(-mt)dt = \frac{1}{m} \left[1 - \exp(-mt)\right] \tag{25}
\]

and

\[
T = t + \frac{\chi^2}{2m} \left[1 - \exp(-2mt)\right] - \frac{2\chi}{m} \left[1 - \exp(-mt)\right] \tag{26}
\]

The old time variable \(t\) is eliminated from equations (25) and (26) to get:

\[
T = (1 - \chi)^2 T^* \tag{27}
\]
This relationship is used in equation (23) to obtain equation (21). To get it, the unsteady flow parameter, \(m\) is considered much smaller than 1.0, so its second and higher degree terms in any exponential or logarithmic or binomial expansions are omitted. Also, while choosing an expression of \(f(mt)\), it is ascertained that through the transformation (18), we get \(T = 0\) for \(t = 0\), so that the nature of the initial condition does not change in this time domain.

Now the Laplace transform may be used to get the analytical solution. But to use it more conveniently, the convective term in the advection–diffusion equation (19) is eliminated by a transformation:

\[
C(Z, T) = K(Z, T) \exp \left( \frac{u_0}{2D_0} Z - \frac{u_0^2}{4D_0} T \right) \tag{28}
\]

Thus, the initial and boundary value problem defined by equations (19)–(22) may be written in terms of a new dependent variable, \(K(Z, T)\):

\[
\frac{\partial K}{\partial T} = D_0 \frac{\partial^2 K}{\partial Z^2} \tag{29}
\]

\[
K(Z, T) = 0 \quad (T = 0, Z \geq 0) \tag{30}
\]

\[
K(Z, T) = C_0 \exp \left\{ \left( au_0\gamma + \frac{u_0^2}{4D_0} \right) T \right\} \tag{31}
\]

\[(Z = 0, T > 0)\]

and

\[
\frac{\partial K}{\partial Z} = - \frac{u_0}{2D_0} K \quad (Z \to \infty, T \geq 0) \tag{32}
\]

Now, applying the technique of Laplace transformation, and using the necessary transformations backwards, the desired solution may be obtained as equation (33).

\[
\frac{c(x, t)}{C_0} = \frac{1}{2} \left[ \exp \left\{ \left( \frac{u_0}{2D_0} - \frac{\mu}{\sqrt{D_0}} \right) \frac{f_1(mt)}{af(mt)} \ln(1 + ax) \right\} \right. \text{erfc} \left( \frac{f_1(mt)/f(mt)}{2a(1 - \lambda)\sqrt{D_0T^*}} \ln(1 + ax) - \mu(1 - \lambda)\sqrt{T^*} \right) + \left. \exp \left\{ \left( \frac{u_0}{2D_0} + \frac{\mu}{\sqrt{D_0}} \right) \frac{f_1(mt)}{af(mt)} \ln(1 + ax) \right\} \right] \text{erfc} \left( \frac{f_1(mt)/f(mt)}{2a(1 - \lambda)\sqrt{D_0T^*}} \ln(1 + ax) + \mu(1 - \lambda)\sqrt{T^*} \right) \tag{33}
\]

where \(f_1(mt) = 1 - (aD_0/u_0)f(mt)\), \(\gamma = (1 - aD_0/ u_0)^{-2}\), \(\mu = \sqrt{au_0\gamma + u_0^2/(4D_0)}\), and by using equation (8), \(T^*\) may be expressed in terms of \(t\) for an expression of \(f(mt)\). It can be verified that equation (27), obtained for \(f(mt) = \exp(-mt)\), may also be obtained for an exponentially accelerating function \(f(mt) = \exp(mt)\). So this solution is applicable for both the expressions of \(f(mt)\). Also, for \(m = 0\), solution (33) reduces to a solution of a previous work (Kumar et al. 2010).

### 3 RESULTS AND DISCUSSION

Concentration values are evaluated from analytical solution (33) for \(C_0 = 1.0\) initial velocity \(u_0 = 0.60\) (km/year), and initial dispersion coefficient \(D_0 = 0.71\) (km²/year), heterogeneity parameter \(a = 0.1\) (km⁻¹) and unsteady flow parameter \(m = 0.1\) (year⁻¹). The concentration values \((C/C_0)\) are evaluated in a finite domain \(0 \leq x \leq 1\) of the semi-infinite medium. Exponentially accelerating and decelerating forms of \(f(mt)\) are considered. The numerical values depicting the velocity distribution in the assumed domain at \(t = \text{year} = 0.1, 0.4, 0.7\) and 1.0, for the forms: (i) \(u = u_0 \exp(mt)(1 + ax)\), and (ii) \(u = u_0 \exp(-mt)(1 + ax)\), are arranged in Table 1. The first row at \(t = 0\) shows the increasing variation in steady velocity, and other rows show the same for unsteady velocities defined by either of the above two forms. It may be observed that in the case of (i), velocity, at a particular position (in a particular column), increases with time, while it decreases in case (ii). Each variation in velocity, whether due to heterogeneity (in a particular row) or due to unsteadiness (in a particular column), is of small order, which is the actual occurrence in most natural cases. It ensures that Fick’s law (on which ADE is based) holds good in spite of the heterogeneity of the medium and unsteadiness of the flow field.

For \(f(mt) = \exp(-mt)\), the concentrations evaluated from solution (33) for \(t = \text{year} = 0.1, 0.4, 0.7\) and 1.0 are depicted by four solid curves in
Table 1  Distribution of velocity in the domain $0 \leq x \leq 1$ for two cases: (i) $u = u_0 \exp(mt)(1 + ax)$, and (ii) $u = u_0 \exp(-mt)(1 + ax)$, for $m = 0.1 \text{ d}^{-1}$, $a = 0.1 \text{ km}^{-1}$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>0.2</th>
<th>0.4</th>
<th>0.6</th>
<th>0.8</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t = 0$</td>
<td>0.600</td>
<td>0.612</td>
<td>0.624</td>
<td>0.636</td>
<td>0.648</td>
<td>0.660</td>
</tr>
<tr>
<td>$t = 0.1$</td>
<td>(i) 0.606</td>
<td>0.618</td>
<td>0.630</td>
<td>0.642</td>
<td>0.654</td>
<td>0.666</td>
</tr>
<tr>
<td></td>
<td>(ii) 0.594</td>
<td>0.606</td>
<td>0.617</td>
<td>0.629</td>
<td>0.641</td>
<td>0.653</td>
</tr>
<tr>
<td>$t = 0.4$</td>
<td>(i) 0.624</td>
<td>0.636</td>
<td>0.649</td>
<td>0.661</td>
<td>0.674</td>
<td>0.686</td>
</tr>
<tr>
<td></td>
<td>(ii) 0.576</td>
<td>0.588</td>
<td>0.599</td>
<td>0.610</td>
<td>0.622</td>
<td>0.633</td>
</tr>
<tr>
<td>$t = 0.7$</td>
<td>(i) 0.643</td>
<td>0.656</td>
<td>0.669</td>
<td>0.682</td>
<td>0.694</td>
<td>0.707</td>
</tr>
<tr>
<td></td>
<td>(ii) 0.559</td>
<td>0.570</td>
<td>0.581</td>
<td>0.592</td>
<td>0.603</td>
<td>0.615</td>
</tr>
<tr>
<td>$t = 1.0$</td>
<td>(i) 0.663</td>
<td>0.676</td>
<td>0.689</td>
<td>0.702</td>
<td>0.716</td>
<td>0.729</td>
</tr>
<tr>
<td></td>
<td>(ii) 0.542</td>
<td>0.553</td>
<td>0.563</td>
<td>0.572</td>
<td>0.585</td>
<td>0.596</td>
</tr>
</tbody>
</table>

Fig. 1 Solute transport described by the solution (33), and depicted by solid curves, where $f(mt) = \exp(-mt)$, $m = 0.1 \text{ year}^{-1}$ and $a = 0.1 \text{ km}^{-1}$. The only dotted curve represents the numerical solution.

Fig. 2 Comparison of the solution (33), at $t = 1.0 \text{ year}$, for: (i) uniform flow, (ii) exponentially decelerating flow, and (iii) exponentially accelerating flow, where $m = 0.1 \text{ year}^{-1}$ and $a = 0.1 \text{ km}^{-1}$.

Fig. 3 Effect of heterogeneity on solute transport described by solution (33), where $f(mt) = \exp(-mt)$, $m = 0.1 \text{ year}^{-1}$.

Fig. 1. The solute transport behaves as expected, i.e. the concentration values decrease with distance and increase with time. Figure 2 compares the concentrations determined from solution (33) for $t = 1.0 \text{ year}$, for three expressions: (i) $f(mt) = 1$, (ii) $f(mt) = \exp(-mt)$, and (iii) $f(mt) = \exp(mt)$. For the first case, solution (33) reduces to the solution of Kumar et al. (2010). Figure 2 shows that solute transport along exponentially decelerating flow is the slowest, and that along an exponentially accelerating unsteady flow field is fastest. Figure 3 shows the effect of heterogeneity on solute transport along an exponentially decelerating flow field at $t = 0.4 \text{ year}$. The concentration values are evaluated from solution (33) for $a = 0.1, 0.4$ and $0.7 \text{ (km}^{-1})$. It may be observed that at a particular position, concentration decreases with the heterogeneity parameter. In other words, solute transport is slower in a medium of higher heterogeneity of increasing nature than that in a medium of lower heterogeneity. This is because $(u/D)$ at a position decreases with $a$. It may be evaluated from expressions in equations (3a,b) that at $x = 0.5$, the values of $(u/D)$ are 0.83, 0.73 and 0.65, at the above three values of $a$, respectively. Further, in Figs 1–3, the input concentration, i.e. $(c/C_0)$ at $x = 0$, is 1.0 at $t > 0$, which shows that the source is a uniform continuous point source. It may be observed that, in a finite time period, concentration level reaches a stable level. In other words, within a finite period the pollution level of the whole domain reaches a saturated level, hence the source becomes ineffective. So in this finite time domain $f(mt) = \exp(mt)$ may be used.
The ADE in equation (4), along with conditions (5) to (7), is also solved using a finite difference two-level explicit numerical method. The domain \(0 \leq x \leq 1\) is divided into nodes at uniform intervals, \(\Delta x\), as:

\[ x_i = x_{i-1} + \Delta x; \quad i = 2, 3, 4, \ldots, 1, \quad x_1 = 0, x_1 = 1.0 \]

Similarly the \(t\)-domain is divided in to equally spaced nodes:

\[ t_j = t_{j-1} + \Delta t; \quad j = 2, 3, 4, \ldots, 1, \quad t_1 = 0 \]

The concentration at \((x_i,t_j)\) is assumed to be \(C_{i,j}\) and we use following finite-difference two-level explicit approximations:

\[
\frac{\partial c}{\partial t} \approx \frac{c_{i,j+1} - c_{i,j}}{\Delta t}
\]

\[
\frac{\partial c}{\partial x} \approx \frac{c_{i+1,j} - c_{i,j}}{\Delta x}
\]

and

\[
\frac{\partial^2 c}{\partial x^2} \approx \frac{c_{i+1,j} - 2c_{i,j} + c_{i-1,j}}{(\Delta x)^2}
\]

Using these in equation (4), the unknown value \(c_{i,j+1}\) is evaluated subject to the following conditions:

\[
c_{i,1} = 0; \quad i = 1, 2, 3, 4, \ldots, 1
\]

\[
c_{1,j} = C_0; \quad j \geq 2
\]

\[
c_{i+1,j} = c_{i,j}; \quad i = 1, 2, 3, 4, \ldots, 1 - 1 \text{ and } j \geq 2
\]

The computations are performed until \(t = 0.1\) year, by choosing \(\Delta x = 0.1\) and \(\Delta t = 0.001\). The stability condition \(\left( \frac{\Delta t}{(\Delta x)^2} \right) \leq \frac{1}{2}\), is satisfied. The result is depicted by the dotted curve in Fig. 1, which indicates that the analytical and numerical solutions are in very good agreement. This validates the analytical solution and the mathematical procedure by which it has been obtained.

4 CONCLUSIONS

In the works of Yates (1990), Logan (1996) and Zoppou and Knight (1997), the dispersion coefficient was also considered as spatially dependent but to go on increasing as \(x\) increases along the semi-infinite domain; hence its limiting value was assumed. In the present work, the changes due either to heterogeneity or to unsteadiness can be managed so as to remain small or of a desired order, by choosing appropriate values of their respective parameters, \(a\) and \(m\). In addition, a function for the velocity may be interpolated to describe heterogeneity of decreasing order. Though the analytical treatment has restricted the description of the heterogeneity of the medium to a linear function, quadratic or sinusoidal expressions may be chosen when dealing with the problem numerically. A new space variable introduced through the transformation in equation (17) serves as a moving coordinate. It is like the one transformation \(Z = x - ut\) (Bear 1972), which eliminates the convective term from the ADE with constant coefficient. The effects of heterogeneity and unsteadiness are studied and illustrated. More expressions for \(f(m,t)\) may be chosen, but for one such expression an explicit relationship between the new time variables \(T^*\) and \(T\), such as that given by equation (27), should occur. Thus an analytical solution of ADE with variable coefficients, describing solute transport from the perspective of Schiedegger’s (1957) theory has been obtained by using the Laplace Integral Transform Technique (LITT).

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