SOLUTIONS OF A SYSTEM OF FORCED BURGERS EQUATION IN TERMS OF GENERALIZED LAGUERRE POLYNOMIALS∗

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Abstract In this article, we obtain explicit solutions of a linear PDE subject to a class of radial square integrable functions with a monotonically increasing weight function \( |x|^n-1 e^{\beta |x|^2/2} \), \( \beta \geq 0, x \in \mathbb{R}^n \). This linear PDE is obtained from a system of forced Burgers equation via the Cole-Hopf transformation. For any spatial dimension \( n > 1 \), the solution is expressed in terms of a family of weighted generalized Laguerre polynomials. We also discuss the large time behaviour of the solution of the system of forced Burgers equation.

Key words forced Burgers equation; radial Hermite functions; generalized Laguerre polynomials; self-similar solutions

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1 Introduction

Nonlinear evolution equations (NLEE) of the form

\[
 u_t + a(x,t)uu_x + b(x,t)u_{xx} = f(x,t), \quad x \in \mathbb{R}, \ t > 0,
\]

where \( a, b \) and \( f \) are real-valued functions, are commonly referred as forced or nonhomogeneous Burgers equations. Here \( f \) is the forcing or nonhomogeneous term. Xu et al. [1] presented a generalized version of the Cole-Hopf transformation for linearizing a fairly large class of NLEE (1.1) to the well known Heat equation. The generalized Cole-Hopf transformation constructed by Xu et al. [1], using symbolic computation, is of the form

\[
 u(x,t) = \alpha(x,t) \frac{\phi_\xi(\xi, \tau)}{\phi_\xi(\xi, \tau)} + \beta(x,t), \quad \xi = \xi(x,t), \ \tau = \tau(t),
\]

where \( \alpha(x,t) \neq 0, \xi_x \neq 0 \). The functions \( \alpha, \beta, \xi \) and \( \tau \) are determined in terms of the functions \( a, b, f \) appearing in NLEE (1.1), and some arbitrary functions and arbitrary constants. Furthermore, the functions \( a, b \) and \( f \) are required to satisfy some constraint conditions under which the transformation (1.2) linearizes the constrained class of NLEE (1.1) to the Heat equation

\[
 \phi_\tau - \phi_{\xi \xi} = 0.
\]

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Forced Burgers equations (1.1) appear in modelling many physical and biological phenomena (see for instance, [1]–[4] and the references therein). The forced Burgers equation (1.4) corresponds to the Burgers turbulence problem with quadratic external potential. Barndorff-Nielsen and Leonenko [3] studied the Burgers turbulence problem with linear or quadratic external potential. Ding et al. [5] obtained an infinite series solution of a nonhomogeneous Burgers equation using eigen function expansion technique. In a series of papers, Rao and Yadav [6]–[8] studied the scalar nonhomogeneous Burgers equation
\[ u_t + uu_x = u_{xx} + \frac{kx}{(2\beta t + 1)^2}, \quad x \in \mathbb{R}, \quad t > 0, \]  
subject to certain classes of bounded and unbounded initial data. Recently, Yadav [9] studied initial value problems for a system of forced Burgers equations. Subject to various classes of bounded and unbounded initial data, Yadav [9] expressed the solution of the initial value problems in terms of infinite series of scaled multidimensional Hermite functions. We refer Kloosterziel [10] for the initial work done in this direction for the Heat equation. In this paper, we carry out a similar study for the same system of forced Burgers equations subject to a class of initial data possessing radial symmetry. Here the solutions are expressed in terms of generalized Laguerre polynomials which appear in a family of self-similar solutions of the system of forced Burgers equation. We refer Yadav [9] for a detailed account on the motivation for the present work. At this point we would also like to make a reference to the works of Bluman and Cole [11] on general similarity solutions of the Heat equation, and Doyle and Englefield [12] on similarity solutions of a generalized Burgers equation.

Let \( U : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}^n, \ n > 1 \) be the \( n \)-dimensional vector \((u_1, u_2, \ldots, u_n)\). Then, we consider an equation analogous to the nonhomogeneous Burgers equation (1.4) for \( U \)
\[ U_t + U \cdot \nabla U = \Delta U + \frac{kx}{(2\beta t + 1)^2}, \quad x \in \mathbb{R}^n, \quad t > 0, \]  
where \( k > 0 \) and \( \beta \geq 0 \) are constants and \( x \) is the \( n \)-tuple \((x_1, x_2, \ldots, x_n)\). Equation (1.5) may be deduced from the Navier-Stokes equation as a special case (see for instance, Neklyudov [13], Young et al. [14]). The term \( U \cdot \nabla U \) in (1.5) appears in the Navier-Stokes equation also and is referred as convective acceleration. The convective acceleration term is interpreted as
\[ U \cdot \nabla U = \frac{1}{2} \nabla(|U|^2) + (\nabla \times U) \times U. \]  
We study equation (1.5) subject to the initial condition
\[ U(x, 0) = U_0(x) = (u_{01}(x), u_{02}(x), \ldots, u_{0n}(x)), \quad x \in \mathbb{R}^n. \]  
The Cole–Hopf transformation (see Joseph and Sachdev [15])
\[ U(x, t) = -\gamma \nabla \phi(x, t)/\phi(x, t) \]  
transforms (1.5)–(1.7) to
\[ \phi_t = \Delta \phi - \frac{k|x|^2}{4(2\beta t + 1)^2} \phi, \quad x \in \mathbb{R}^n, \quad t > 0, \]  
\[ \phi(x, 0) = \Phi_0(x) = \Pi_{i=1}^n \exp\left( -\frac{1}{2} \int_{x_i}^{x_i} u_0(y) dy \right), \quad x \in \mathbb{R}^n, \]
where $u_{0i}$ as in (1.7) and $\prod_{i=1}^{n}a_i = a_1a_2\cdots a_n$. For linearization of systems of Burgers' equation by the vector form of Cole-Hopf transformation one may refer to the works of Joseph [16] and Joseph and Vasudeva Murthy [17].

We may note that in view of transformation (1.8),
\[ \nabla \times U = 0, \quad \nabla \times U_0 = 0, \] (1.11)
so that, (1.6) reduces to
\[ U \cdot \nabla U = \frac{1}{2} \nabla (|U|^2). \] (1.12)
Condition (1.11) renders the solution $U$ of the forced Burgers equation (1.5) to be vorticity free. Vorticity free solutions of forced Burgers equation have a wide range of applications in statistical physics, for instance, cosmology, vehicular traffic, directed polymers in random media etc., see Bec and Khanin [2].

The transformation vector
\[ U(x, t) = \alpha(x, t) \frac{\nabla \phi(\xi, \tau)}{\phi(\xi, \tau)} + \eta(x, t), \] (1.13)
where
\[ \alpha(x, t) = \alpha(t) = -2(2\beta t + 1)^{2A/\beta}, \quad \eta(x, t) = -4Ax(2\beta t + 1)^{-1}, \]
\[ \xi(x, t) = x(2\beta t + 1)^{2A/\beta}, \quad \tau(t) = \lambda[(2\beta t + 1)^{1/(A\beta)} - 1], \]
\[ \lambda = 1/(8A + 2\beta), \quad A = (-\beta + \sqrt{\beta^2 + k})/4, \]
transforms the system of forced Burgers equation (1.5) to the Heat equation
\[ \phi_\tau = \Delta \phi \]
(see Yadav [9] for more details). The transformation vector (1.13) is in accordance with the generalized Cole–Hopf transformation (1.2) presented in Xu et al. [1] and satisfies the constraint conditions (22), (23) and (26) derived there.

In this paper, we consider a class of initial data $U_0$ such that $\Phi_0$ is radial and
\[ \Phi_0(x) \in L^2_{\text{rad}}(\mathbb{R}^n, |x|^{n-1}e^{\beta|x|^2/2}), \quad \beta \geq 0, \ n > 1. \] (1.14)
In Section 2, we show that for any spatial dimension $n > 1$, the solution of the Cauchy problem (1.9)–(1.10) with $\Phi_0$ satisfying (1.14) can be expressed in terms of a family of weighted generalized Laguerre polynomials.

Rao and Yadav [6], [8] obtained explicit solution of the initial value problem (1.9)–(1.10) with $n = 1$. They considered two classes of initial data $L^2(\mathbb{R}, e^{\beta x^2/2}), \beta > 0$ and $L^2(\mathbb{R}, e^{-A_1|x|^2/2})$, where $A_1$ is a specific constant.

With $\Phi_0(x) \in L^2(\mathbb{R}, e^{\beta x^2/2})$, the solution of the 1-d problem (1.9)–(1.10) may be written as
\[ \phi(x, t) = \sum_{n=0}^{\infty} c_n \phi_n(x, t), \quad c_n = \int_{-\infty}^{\infty} \Phi_0(x) \exp(\beta x^2/4) \psi_n(\sqrt{\beta}x) dx. \] (1.15)
Here $\{\phi_n(x, t)\}_{n=0}^{\infty}$ is a family of self-similar solutions of the one dimensional equation (1.9) and is given by
\[ \phi_n(x, t) = b^{-\alpha_n/\beta}(t) \exp(-\beta \eta^2/4) \psi_n(\sqrt{\beta} \eta), \quad \eta = \frac{x}{b(t)}, \ n = 0, 1, 2, \cdots, \] (1.16)
where \( b(t) = \sqrt{2\beta t + 1} \), \( \mu = \frac{1}{2} \sqrt{\beta^2 + k} \), \( \alpha_n = (2n + 1)\mu + \frac{\mu}{2} \), \( n = 0, 1, 2, \cdots \), and
\[
\psi_n(\sqrt{\mu}\eta) = \frac{1}{k_n} e^{-\mu\eta^2/2} H_n(\sqrt{\mu}\eta), \quad n = 0, 1, 2, \cdots.
\] (1.17)

Here \( k_n^2 = 2^n n! \sqrt{\pi/\mu} \), \( H_n \) is the Hermite polynomial of order \( n \) and the Hermite polynomials are defined by the Rodrigues’ formula
\[
H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}, \quad n = 0, 1, 2, \cdots.
\]
The functions \( \psi_n(\sqrt{\mu}\eta) \), \( n = 0, 1, 2, \cdots \) satisfy an orthonormality relation
\[
\int_{\mathbb{R}} \psi_n(\sqrt{\mu}\eta) \psi_m(\sqrt{\mu}\eta) d\eta = \delta_{nm}, \quad n, m = 0, 1, 2, \cdots,
\]
where \( \delta_{nm} \) is the Kronecker delta. Further the family of functions \( \{\psi_n(\sqrt{\mu}\eta)\}_{n=0}^{\infty} \) forms an orthonormal basis for \( L^2(\mathbb{R}) \), see Higgins [18].

The organization of this article is as follows. Section 2 deal with the class of initial data \( \Phi_0 \) satisfying (1.14). Here a family of self-similar solutions involving generalized Laguerre polynomials and possessing radial symmetry and orthogonality properties is constructed for the linearized damped heat equation (1.9). We then express the solution of the corresponding initial value problem for the forced Burgers equation (1.5) and finally obtain its large time asymptotic behaviour. In Section 3, we express the solution of the 3-dimesional Cauchy problem (1.9)–(1.10) in terms of a weighted radial Hermite functions family. In the Appendix, we present a second family of self-similar solutions for the linearized damped heat equation (1.9). Section 4 presents the conclusion of this work.

2 Self-Similar Solutions of (1.9) in Terms of Generalized Laguerre Polynomials

In this section, we assume that the initial profile \( \Phi_0(x) \) satisfies condition (1.14). In other words, \( \Phi_0 \) is radially symmetric and square integrable function on \( \mathbb{R}^n \) with respect to the weight function \( |x|^{n-1} e^{\beta|x|^2/2} \). We show that the solution of the Cauchy problem (1.9)–(1.10) can be expressed in terms of weighted generalized Laguerre polynomials.

First, we obtain families of self-similar solutions of (1.9) in terms of generalized Laguerre polynomials. We consider self-similar solutions of (1.9) of the form
\[
\phi(x, t) = \frac{1}{a(t)} f(\eta), \quad \eta = \frac{|x|}{b(t)},
\] (2.1)
here \( a(t) \) and \( b(t) \) are such that \( a(0) = b(0) = 1 \). Then
\[
\nabla \phi = \frac{f'(\eta)}{a(t)b(t)} \frac{x}{|x|} = \frac{x}{|x|} \frac{\partial \phi}{\partial x}.
\] (2.2)

Now substituting (2.1) in (1.9), we get
\[
f''(\eta) + n \frac{1}{\eta} f'(\eta) + b(t)b'(t) f'(\eta) + \frac{b^2(t)a'(t)}{a(t)} f(\eta) - \frac{kb^4(t)}{4(2\beta t + 1)^2} \eta^2 f(\eta) = 0.
\] (2.3)

The functions \( a(t) \) and \( b(t) \) will be determined in such a way that (2.3) is free of the time dependent coefficients, that is,
\[
b(t)b'(t) = \tilde{\beta}_1, \quad \frac{b^2(t)a'(t)}{a(t)} = \alpha, \quad b^4(t) = (2\beta t + 1)^2.
\] (2.4)
Here $\alpha$ and $\tilde{\beta}_1$ are some constants to be determined. Integrating the first two relations in (2.4) subject to $a(0) = b(0) = 1$ and making use of the third relation in (2.4), we arrive at $\tilde{\beta}_1 = \beta$ and

\begin{align*}
a(t) &= b^{\alpha/\beta}(t), \quad b(t) = \sqrt{2\beta t + 1}, \quad \text{if } \beta > 0; \\
a(t) &= e^{\alpha t}, \quad b(t) = 1, \quad \text{if } \beta = 0.
\end{align*}

(2.5)

(2.6)

We discuss the cases $\beta > 0$ and $\beta = 0$ jointly.

**Case I** $\beta \geq 0$

We use (2.5) or (2.6) in (2.3) to obtain

\[ f''(\eta) + \left( n - 1 + \beta \eta \right) f'(\eta) + \alpha f(\eta) - \frac{k}{4} f(\eta) = 0, \quad \beta \geq 0. \]

(2.7)

The transformation

\[ \xi = \eta^2, \quad g(\xi) = f(\eta) \]

transforms (2.7) with $\beta \geq 0$ to

\[ 4\xi g'' + 2(\beta \xi + n) g' + (\alpha - \frac{k}{4}\xi)g = 0, \quad \beta \geq 0. \]

(2.9)

An another transformation

\[ g(\xi) = e^{\frac{h}{4}} z(\zeta), \quad \zeta = \frac{\xi}{2} \sqrt{\beta^2 + k}, \quad h = \frac{1}{4}(\beta + \sqrt{\beta^2 + k}), \quad \beta \geq 0 \]

(2.10)

(see Polyanin and Zaitsev [19]) transforms (2.9) to a degenerate hypergeometric differential equation

\[ \zeta z'' + (b - \zeta) z' - az = 0. \]

(2.11)

The parameters $a$, $b$ are given as

\[ a = \frac{n(\beta + \sqrt{\beta^2 + k}) - 2\alpha}{4\sqrt{\beta^2 + k}}, \quad b = \frac{n}{2}, \quad \beta \geq 0. \]

(2.12)

If we choose the parameter $\alpha$ in the expression for $a$ as

\[ \alpha = \alpha_{m_1} = \frac{1}{2} n \beta + (n + 4m_1) \sqrt{\beta^2 + k}, \quad \beta \geq 0, \quad m_1 = 0, 1, 2, \ldots, \]

(2.13)

then $a = -m_1$ and the solutions of (2.11) are given by the generalized Laguerre polynomials $L_{m_1}^{(b-1)}(\zeta)$.

A generalized Laguerre polynomial $L_n^{(\alpha)}(x)$ for $\alpha > -1$ is defined by the Rodrigues’ formula

\[ L_n^{(\alpha)}(x) = \frac{x^{-\alpha} e^x}{n!} \frac{d^n}{dx^n} (e^{-x} x^{n+\alpha}), \quad \alpha > -1, \quad n = 0, 1, 2, \ldots \]

(2.14)

and satisfies the orthogonality relation

\[ \int_0^\infty x^\alpha e^{-x} L_n^{(\alpha)}(x) L_m^{(\alpha)}(x) dx = \frac{\Gamma(n + \alpha + 1)}{n!} \delta_{nm}, \]

(2.15)

where $\delta_{nm}$ is the Kronecker delta. We may note that the normalized sequence of generalized Laguerre polynomials

\[ \hat{k}_n(\alpha) L_n^{(\alpha)}(x), \quad \hat{k}_n(\alpha) = \sqrt{\frac{n!}{\Gamma(n + \alpha + 1)}}, \quad n = 0, 1, 2, \ldots \]

(2.16)

forms an orthonormal basis of $L^2(\mathbb{R}^+, x^\alpha e^{-x})$. 


In view of transformations (2.8) and (2.10), we obtain the following family of self-similar solutions of (1.9)
\[
\phi_m(x, t) = k_m \left( \frac{n-2}{2} \right) b^{-\alpha_m} \beta(t) \exp(h \eta^2) L_{m_1}^{(2)} \left( \frac{\eta^2}{2} \sqrt{\beta^2 + k} \right), \quad \text{if } \beta > 0,
\]
(2.17)
\[
\phi_m(x, t) = k_m \left( \frac{n-2}{2} \right) e^{-\sqrt{\kappa(n+4m_1)} t/2} e^{-\sqrt{\kappa|x|^2/4}^{(\frac{2}{2})}} \left( \frac{\sqrt{\kappa|x|^2}}{2} \right), \quad \text{if } \beta = 0,
\]
(2.18)
where \( m = 1, 2, \ldots \) and \( h \) is as in (2.10). The constants \( k_m (m - 2)/2 \) are defined as
\[
k_m \left( \frac{n-2}{2} \right) = 2^{(2-n)/4} (\beta^2 + k)^{n/8} \tilde{k}_m \left( \frac{n-2}{2} \right), \quad \beta \geq 0, \quad m = 1, 2, \ldots,
\]
(2.19)
here the constants \( \tilde{k}_m (\cdot) \) are obtained from (2.16). In view of the definition (2.14) and the fact that \( n \geq 1 \) (spatial dimension), the generalized Laguerre polynomials \( L_{m_1}^{(2)}(\zeta) \) are defined for \( n \geq 1 \). We may note that the sequence \( \{ \phi_m(x, 0) \} \) forms an orthonormal basis of the function space \( L^2(\mathbb{R}^+, |x|^{n-1} e^{\beta|x|^2/2} d|x|) \), \( \beta \geq 0 \) for all \( n \geq 1 \).

If the initial data \( U_0(x) \) is such that \( \Phi_0(x) \in L^2_{rad}(\mathbb{R}^n, |x|^{n-1} e^{\beta|x|^2/2}) \), \( \beta \geq 0 \) for \( n \geq 1 \) then \( \Phi_0(x) \) may be written as
\[
\Phi_0(x) = \sum_{m=1}^{\infty} c_m \phi_m(x, 0), \quad c_m = \int_{\mathbb{R}^+} \Phi_0(x) \phi_m(x, 0) |x|^{n-1} e^{\beta|x|^2/2} d|x|,
\]
(2.20)
here \( \phi_m(x, 0) \) are obtained from (2.17) or (2.18) according as \( \beta > 0 \) or \( \beta = 0 \). The infinite series in (2.20) converges to \( \Phi_0(x) \) in \( L^2_{rad}(\mathbb{R}^n, |x|^{n-1} e^{\beta|x|^2/2}) \), \( \beta \geq 0 \). One may note that the function spaces \( L^2_{rad}(\mathbb{R}^n, |x|^{n-1} e^{\beta|x|^2/2}) \) and \( L^2(\mathbb{R}^+, |x|^{n-1} e^{\beta|x|^2/2} d|x|) \) are identified with each other. The solution of the Cauchy problem (1.9)–(1.10) subject to the condition
\[
\Phi_0(x) \in L^2_{rad}(\mathbb{R}^n, |x|^{n-1} e^{\beta|x|^2/2}), \quad \beta \geq 0
\]
may be written as
\[
\phi(x, t) = \sum_{m=1}^{\infty} c_m \phi_m(x, t),
\]
(2.21)
where \( \phi_m(x, t) \) are given by (2.17) or (2.18) according as \( \beta > 0 \) or \( \beta = 0 \).

**Remark 2.1** Let \( \beta \geq 0 \) and \( \mathbb{R}^+ := \mathbb{R}^n \times [0, T] \) for any \( T > 0 \). Further let \( \{ S_N \Phi \} \) denote the sequence of partial sums of the series (2.21). Then \( \{ S_N \Phi \} \) converges to \( \phi \) in \( L^2_{rad}(\mathbb{R}^+, \eta^{n-1} e^{\beta \eta^2/2}) \) as \( N \to \infty \).

### 2.1 Large Time Behaviour

Now we discuss the large time behavior of solution of the Cauchy problem for the system of forced Burgers equation (1.5). From (2.13) and (2.17), it is easy to see that
\[
\frac{\phi_{m_1+1}}{\phi_{m_1}} = C(m_1, n) b^{-4\mu/\beta(t)} \left( \frac{L_{m_1}^{(2)}(\eta^2 \mu)}{L_{m_1}^{(2)}(\eta^2 \mu)} \right) \to 0 \quad \text{as } t \to \infty \text{ and } \eta \in O(1),
\]
where \( C(m_1, n) = \sqrt{2(m_1+1)/(2m_1 + n)} \). Therefore,
\[
\phi(x, t) \sim c_{M_1} \phi_{M_1}(x, t) \quad \text{as } t \to \infty \text{ and } |x| \in O(\sqrt{t}),
\]
(2.22)
here \( M_1 \) is the least value of \( m_1 \) such that \( c_{M_1} \neq 0 \). Now from (2.2), we have
\[
\nabla \phi = \frac{\phi}{|x|} \sum_{m_1=0}^{\infty} c_m \phi_{m_1}(x, t),
\]
(2.23)
where $\partial_{x}\phi_{m_{1}}$ is obtained by differentiating (2.17) with respect to $|x|$ as

$$\partial_{x}\phi_{m_{1}} = 2k_{m_{1}}\left(\frac{n}{2} - \frac{n - 2}{2}\right)b^{-\left(\alpha_{m_{1}} + \beta_{m_{1}}\right)}(t)\eta h^{n}2\left(\mu_{\frac{d}{d\xi}}L_{m_{1}}^{(1)}(\xi) + hL_{m_{1}}^{(1)}(\xi)\right), \tag{2.24}$$

disregarding the cases $\beta_{m_{1}} = 0$. Once again it is easy to see that

$$\frac{\partial_{x}\phi_{m_{1} + 1}}{\partial_{x}\phi_{m_{1}}} = C(m_{1}, n)b^{-4\nu/\beta}(t)\frac{\mu_{\frac{d}{d\xi}}L_{m_{1} + 1}^{(1)}(\xi) + hL_{m_{1} + 1}^{(1)}(\xi)\mu_{\frac{d}{d\xi}}L_{m_{1}}^{(1)}(\xi) + hL_{m_{1}}^{(1)}(\xi)\rightarrow 0 \text{ as } t \to \infty \text{ and } \eta \in O(1).$$

Therefore,

$$\nabla \phi \sim \frac{x}{|x|}c_{M_{1}}\partial_{x}|\phi_{M_{1}}(x, t)\text{ as } t \to \infty \text{ and } |x| \in O(\sqrt{t}), \tag{2.25}$$

where $M_{1}$ is the least value of $m_{1}$ such that $c_{M_{1}} \neq 0$. From (2.22), (2.25) and the Cole-Hopf transformation (1.8) one can easily obtain the large time behaviour of the solution of Cauchy problem for the forced Burgers equation (1.5) as

$$U(x, t) \sim -2\frac{x\partial_{x}\phi_{0}(x, t)}{|x|\phi_{0}(x, t)} \text{ as } t \to \infty \text{ and } |x| \in O(\sqrt{t}). \tag{2.26}$$

Let $M_{1} = 0$, then $c_{0} \neq 0$ and

$$-2\frac{x\partial_{x}\phi_{0}(x, t)}{|x|\phi_{0}(x, t)} = -4hb^{-2}(t)x = \frac{A_{0}x}{2\beta t + 1},$$

where $A_{0} = \beta + \sqrt{\beta^{2} + k}$. Thus,

$$U(x, t) \sim \frac{A_{0}x}{2\beta t + 1} \text{ as } t \to \infty \text{ and } |x| \in O(\sqrt{t}). \tag{2.27}$$

Following the above discussion and making use of (2.18) one may conclude that for $\beta = 0$,

$$U(x, t) \sim \sqrt{t}x \text{ as } t \to \infty \text{ and } |x| \in O(1). \tag{2.28}$$

3 Self-Similar Solution in Terms of Radial Hermite Functions for $n = 3$

In this section, we show that for $n = 3$ the solution of the Cauchy problem (1.9)–(1.10) may be expressed in terms of a family of weighted radial Hermite functions. For this purpose, we construct a family of self-similar solutions of (1.9). Here the initial data $\Phi_{0}(x)$ is radial and of the class $L^{2}_{\text{rad}}(\mathbb{R}^{3}, |x|^{2}e^{\beta|x|^{2}/2})$, $\beta \geq 0$. From Section 2, one may easily see that self-similar solutions of (1.9) in form (2.1) leads to the ODE (2.7). We individually discuss the cases $\beta > 0$ and $\beta = 0$, one after another.

**Case I $\beta > 0$**

Introducing the transformation

$$f(\eta) = \exp\left(-\frac{\beta}{4}\eta^{2} - \frac{n - 1}{2}\log \eta\right)v(\eta) \tag{3.1}$$

in (2.7), we get

$$v''(\eta) + \left[\frac{(n - 1)(3 - n)}{4}\eta^{-2} + \left(\alpha - \frac{\beta n}{2}\right) - \frac{1}{4}(\beta^{2} + k)\eta^{2}\right]v(\eta) = 0. \tag{3.2}$$
The differential equation (3.2) is not exactly integrable for all values of the parameter \( n \), which represents the dimension of the forced Burgers system. Only for \( n = 1 \) or \( n = 3 \), equation (3.2) is exactly integrable. Here we are interested in the higher dimensional problem, therefore, we consider ODE (3.2) for \( n = 3 \). Since \( \beta^2 + k > 0 \), we set

\[
\mu = \frac{1}{2} \sqrt{\beta^2 + k}, \quad \alpha_m = (2m + 1)\mu + \frac{3\beta}{2}, \quad m = 0, 1, 2, \ldots. \tag{3.3}
\]

One may note that for \( n = 3 \) and \( \alpha = \alpha_m, \ m = 0, 1, 2, \ldots \), the solutions of (3.2) are the family of Hermite functions

\[
\psi_m(\sqrt{\mu} \eta) = \frac{1}{k_m} e^{-\mu \eta^2/2} H_m(\sqrt{\mu} \eta), \quad m = 0, 1, 2, \ldots. \tag{3.4}
\]

Here \( \mu \) is as defined in (3.3), \( k_m^2 = 2^m m! \sqrt{\pi} / \mu \) and \( H_m \) is the Hermite polynomial of order \( m \) (see Rao and Yadav [6]). Thus for \( n = 3 \), we obtain a family of self-similar solutions of (1.9) as

\[
\phi_m(x, t) = b^{-\alpha_m - \beta}(t) \eta^{-1} \exp(-\beta \eta^2 / 4) \psi_m(\sqrt{\mu} \eta), \quad m = 0, 1, 2, \ldots. \tag{3.5}
\]

It is easy to see that the sequence \( \{ \phi_m(x, 0) \}_{m=0}^{\infty} \) is an orthonormal basis of \( L^2_{\text{rad}}(\mathbb{R}^3, |x|^2 e^{\beta |x|^2 / 2}) \).

Thus if \( \Phi_0(x) \in L^2_{\text{rad}}(\mathbb{R}^3, |x|^2 e^{\beta |x|^2 / 2}) \), then we have the expansion

\[
\Phi_0(x) = \sum_{m=0}^{\infty} c_m \phi_m(x, 0) = \sum_{m=0}^{\infty} c_m |x|^{-1} \exp(-\beta |x|^2 / 4) \psi_m(\sqrt{\mu} |x|), \tag{3.6}
\]

where \( c_m, \ m = 0, 1, 2, \ldots \) are determined by the formula

\[
c_m = \int_0^\infty \Phi_0(x) |x| \exp(\beta |x|^2 / 4) \psi_m(\sqrt{\mu} |x|) |x| \, dx, \quad m = 0, 1, 2, \ldots. \tag{3.7}
\]

The solution \( \phi(x, t) \) of (1.9)–(1.10) is given by

\[
\phi(x, t) = \sum_{m=0}^{\infty} c_m \phi_m(x, t), \tag{3.8}
\]

where \( c_m \) is as in (3.7).

Now we discuss the large time behavior of solution of the Cauchy problem for the system of forced Burgers equation (1.5) when \( n = 3 \) and \( \beta > 0 \). From (3.3) and (3.5), it is easy to see that

\[
\frac{\phi_{m+1}}{\phi_m} = b^{-2 \mu / \beta}(t) \frac{\psi_{m+1}(\sqrt{\mu} \eta)}{\psi_m(\sqrt{\mu} \eta)} \to 0 \quad \text{as} \quad t \to \infty \quad \text{and} \quad \eta \in O(1).
\]

Therefore,

\[
\phi(x, t) \sim c_M \phi_M(x, t) \quad \text{as} \quad t \to \infty \quad \text{and} \quad |x| \in O(\sqrt{t}), \tag{3.9}
\]

here \( M \) is the least value of \( m \) such that \( c_M \neq 0 \). Now from (2.2), we have

\[
\nabla \psi = \frac{x}{|x|} \sum_{m=0}^{\infty} c_m \partial_{|x|} \phi_m(x, t), \tag{3.10}
\]

where \( \partial_{|x|} \phi_m \) is obtained by differentiating (3.5) with respect to \( |x| \) as

\[
\partial_{|x|} \phi_m = b^{-(\alpha_m + \beta) / \beta}(t) \eta^{-2} e^{-\beta \eta^2 / 4} \left( \sqrt{\mu \eta} \psi_m'(\sqrt{\mu} \eta) - \frac{\beta \eta^2}{2} \psi_m(\sqrt{\mu} \eta) - \psi_m(\sqrt{\mu} \eta) \right). \tag{3.11}
\]

Once again it is easy to see that

\[
\frac{\partial_{|x|} \phi_{m+1}}{\partial_{|x|} \phi_m} = b^{-2 \mu / \beta}(t) \frac{\sqrt{\mu} \eta \psi_{m+1}(\sqrt{\mu} \eta) - \frac{\beta \eta^2}{2} \psi_{m+1}(\sqrt{\mu} \eta) - \psi_{m+1}(\sqrt{\mu} \eta)}{\psi_m'(\sqrt{\mu} \eta) - \frac{\beta \eta^2}{2} \psi_m(\sqrt{\mu} \eta) - \psi_m(\sqrt{\mu} \eta)}.
\]
\( \nabla \phi \sim \frac{x}{|x|} c_M \partial_x \phi_M (x,t) \) as \( t \to \infty \) and \( |x| \in O(\sqrt{t}) \), \hspace{1cm} (3.12)

Therefore,

\[
\nabla \phi \sim \frac{x}{|x|} c_M \partial_x \phi_M (x,t) \quad \text{as} \quad t \to \infty \quad \text{and} \quad |x| \in O(\sqrt{t}),
\]

here again \( M \) is the least value of \( m \) such that \( c_M \neq 0 \). From (3.9), (3.12) and the Cole-Hopf transformation (1.8) one can easily obtain the large time behaviour of the solution of Cauchy problem for the forced Burgers equation (1.5) as

\[
U(x,t) \sim -2 \frac{x \partial_x \phi_M (x,t)}{|x| \phi_M (x,t)} \quad \text{as} \quad t \to \infty \quad \text{and} \quad |x| \in O(\sqrt{t}).
\] \hspace{1cm} (3.13)

Let \( M = 0 \), then \( c_0 \neq 0 \) and

\[
-2 \frac{x \partial_x \phi_0 (x,t)}{|x| \phi_0 (x,t)} = \frac{A_0 x}{2 \beta + 1} + 2 \frac{x}{|x|^2},
\]

\[
\sim \frac{A_0 x}{2 \beta + 1} \quad \text{as} \quad t \to \infty \quad \text{and} \quad |x| \in O(\sqrt{t}), \quad d > 1,
\]

here \( A_0 = \beta + \sqrt{\beta^2 + k} \) and \( |x| \in \Omega(\sqrt{t}) \) implies \( |x| \geq K_1 t^{d/2} \) for some constant \( K_1 > 0 \) and \( t \) large. Thus,

\[
U(x,t) \sim \frac{A_0 x}{2 \beta + 1} \quad \text{when} \quad K_1 t^{d/2} \leq |x| \leq K_2 t^{1/2} \quad \text{and} \quad t \quad \text{large},
\] \hspace{1cm} (3.14)

here \( K_1 \) and \( K_2 \) are some absolute constants. In the above discussion the symbols \( O \) and \( \Omega \) are Bachmann-Landau asymptotic notations, refer Knuth [20] for definitions.

**Case II** \( \beta = 0 \)

For \( \beta = 0 \), equation (2.7) takes the form

\[
f''(\eta) + \frac{n-1}{\eta} f'(\eta) + \alpha f(\eta) - \frac{k}{4} \eta^2 f(\eta) = 0, \quad \eta = |x|. \] \hspace{1cm} (3.15)

Introducing the transformation

\[
f(|x|) = |x|^{-(n-1)/2} v(|x|)
\] \hspace{1cm} (3.16)

in (3.15), we get

\[
v''(|x|) + \left[ \frac{(n-1)(3-n)}{4} |x|^{-2} + \alpha - \frac{k}{4} |x|^2 \right] v(|x|) = 0.
\] \hspace{1cm} (3.17)

Once again it is easy to see that equation (3.17) is not exactly integrable for all values of the parameter \( n \). Only for \( n = 1 \) or \( n = 3 \), equation (3.17) is exactly integrable. We consider ODE (3.17) for \( n = 3 \). Since \( k > 0 \), we set

\[
\nu = \frac{\sqrt{k}}{2}, \quad \alpha_m = (2m+1) \nu, \quad m = 0, 1, 2, \ldots \]

(3.18)

For \( n = 3 \) and \( \alpha = \alpha_m, \quad m = 0, 1, 2, \ldots \), the solutions of (3.17) are the family of Hermite functions

\[
\psi_m (\sqrt{\nu} |x|) = \frac{1}{k_m} e^{-\nu |x|^2/2} H_m (\sqrt{\nu} |x|), \quad m = 0, 1, 2, \ldots.
\] \hspace{1cm} (3.19)

Here \( \nu \) is as defined in (3.18) and \( k_m^2 = 2^m m! \sqrt{\pi/\nu} \). Thus for \( n = 3 \), we obtain a family of self-similar solutions of (1.9) as

\[
\phi_m (x,t) = e^{-\alpha_m |x|^2} |x|^{-1} \psi_m (\sqrt{\nu} |x|), \quad m = 0, 1, 2, \ldots,
\] \hspace{1cm} (3.20)
where $\alpha_m$ is as in (3.18). It is easy to see that the sequence $\{\phi_m(x,0)\}_{m=0}^{\infty}$ is an orthonormal basis of $L^2(\mathbb{R}^+, |x|^2)$. Thus if $\Phi_0(x) \in L^2(\mathbb{R}^+, |x|^2)$, then we have the expansion

$$
\Phi_0(x) = \sum_{m=0}^{\infty} c_m \phi_m(x, 0) = \sum_{m=0}^{\infty} c_m |x|^{-1} \psi_m(\sqrt{\nu}|x|),
$$

(3.21)

where $c_m$, $m = 0, 1, 2, \ldots$ are determined by the formula

$$
c_m = \int_0^{\infty} \Phi_0(x)|x| \psi_m(\sqrt{\nu}|x|)d|x|, \quad m = 0, 1, 2, \ldots.
$$

(3.22)

The solution $\phi(x, t)$ of (1.9)–(1.10) is given by

$$
\phi(x, t) = \sum_{m=0}^{\infty} c_m \phi_m(x, t),
$$

(3.23)

where $c_m$ is as in (3.22). The least value of $m$, say $M$, such that $c_M \neq 0$ determines the large time behaviour of $\phi(x, t)$. Following the large time asymptotics discussion for the case $\beta > 0$ we conclude that for $\beta = 0$

$$
U(x, t) \sim \sqrt{k}x \quad \text{as} \quad t \to \infty \quad \text{and} \quad k_1 t \leq |x| \leq k_2, \quad c > 0,
$$

(3.24)

where $k_1, k_2$ are some absolute constants.

4 Conclusions

This article is about the large time asymptotics of solution of a Cauchy problem for the system of forced Burgers equation (1.5) on $\mathbb{R}^n \times (0, \infty)$. The Cole-Hopf transformation (1.8) linearizes (1.5) to the linear PDE (1.9). In section 2, we have considered Cauchy problem for (1.9) with radial initial data $\Phi_0 \in L^2_{\text{rad}}(\mathbb{R}^n, |x|^{n-1}e^{\beta|x|^2/2})$. For any dimension $n > 1$, the solution is expressed in terms of weighted generalized Laguerre polynomials. In Section 3, we have expressed the solution of the 3-dimesional Cauchy problem in terms of a family of weighted radial Hermite functions. For each of the various solutions obtained, their large time asymptotic behavior is also analyzed.

References


Appendix

Another family of self-similar solutions of (1.9) involving generalized Laguerre polynomials

The transformation
\[ \xi = \eta^2, \quad g(\xi) = \xi^l f(\eta), \quad l = (n-2)/2 \]  
(A.1)
transforms (2.7) with \( \beta \geq 0 \) to
\[ 4\xi g''' + (2\beta \xi - 2n + 8)g' + (\alpha - \beta(n-2) - \frac{k}{4}\xi)g = 0, \quad \beta \geq 0. \]  
(A.2)
We again refer to Polyanin and Zaitsev [19] for another transformation
\[ g(\xi) = e^{h\xi} z(\zeta), \quad \zeta = \frac{\xi}{2}\sqrt{\beta^2 + k}, \quad h = -\frac{1}{4}(\beta + \sqrt{\beta^2 + k}), \quad \beta \geq 0 \]  
(A.3)
that transforms (A.2) to a degenerate hypergeometric differential equation
\[ \zeta z'' + (d - \zeta)z' - cz = 0. \]  
(A.4)
The parameters \( c, d \) are given as
\[ c = \frac{-(n-4)\sqrt{\beta^2 + k} + \beta n - 2\alpha}{4\sqrt{\beta^2 + k}}, \quad d = -\frac{1}{2}(n-4), \quad \beta \geq 0. \]  
(A.5)
If we choose the parameter \( \alpha \) in the expression for \( c \) as
\[ \alpha = \alpha_m = \frac{1}{2}[n\beta + (4m^2_n - n + 4)\sqrt{\beta^2 + k}], \quad \beta \geq 0, \quad m_n = 0, 1, 2, \cdots, \]  
(A.6)
then \( c = -m^2_2 \) and the solutions of (A.4) are given by the generalized Laguerre polynomials
\[ L^{(d-1)}_{m_2}(\zeta). \]
In view of the transformations (A.1) and (A.3), we obtain the following family of self-similar solutions of (1.9): for $\beta > 0$,
\[
\phi_{m_2}(x,t) = k_{m_2} \left( \frac{2-n}{2} \right) \eta^{2-n} \exp(h\eta^2) L_{m_2}^{(1-\frac{\beta}{2})} \left( \frac{\eta^2}{2} \sqrt{\beta^2 + k} \right), \tag{A.7}
\]
while for $\beta = 0$
\[
\phi_{m_2}(x,t) = k_{m_2} \left( \frac{2-n}{2} \right) |x|^{2-n} e^{-\sqrt{k}(6m_2-2n+8)t+|x|^2)/4} L_{m_2}^{(1-\frac{\beta}{2})} \left( \frac{\sqrt{k} |x|^2}{2} \right), \tag{A.8}
\]
here $m_2 = 0, 1, 2, \cdots$ and $h$ is as in (A.3). The constants $k_{m_2}((2-n)/2)$ are defined as
\[
k_{m_2} \left( \frac{2-n}{2} \right) = 2^{(n-2)/4}(\beta^2 + k)^{(4-n)/4} \tilde{k}_{m_2} \left( \frac{2-n}{2} \right), \quad \beta \geq 0, \quad m_2 = 0, 1, 2, \cdots, \tag{A.9}
\]
where the constants $\tilde{k}_{m_2}(.)$ are obtained from (2.16). In view of definition (2.14) and the fact that $n \geq 1$ (spatial dimension), the generalized Laguerre polynomials $L_{m_2}^{(1-\frac{\beta}{2})}(\zeta)$ are defined for $1 \leq n < 4$. We may note that the sequence $\{\phi_{m_2}(x,0)\}$ also forms an orthonormal basis of the function space $L^2(\mathbb{R}^+, |x|^{n-1} e^{\beta|\eta|^2/2} d\eta)$, $\beta \geq 0$ for $1 \leq n < 4$. 