Solutions of a system of forced Burgers equation

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ABSTRACT

In this article, we obtain explicit solutions of a system of forced Burgers equation subject to some classes of bounded and compactly supported initial data and also subject to certain unbounded initial data. In a series of papers, Rao and Yadav (2010) [1–3] obtained explicit solutions of a nonhomogeneous Burgers equation in one dimension subject to certain classes of bounded and unbounded initial data. Earlier Kloosterziel (1990) [4] represented the solution of an initial value problem for the heat equation, with initial data in $L^2(\mathbb{R}^n, e^{\|x\|^2})$, as a series of self-similar solutions of the heat equation in $\mathbb{R}^n$. Here, we express the solutions of certain classes of Cauchy problems for a system of forced Burgers equation in terms of self-similar solutions of some linear partial differential equations.

1. Introduction

Initial boundary value problems (IBVPs) for second order linear partial differential equations on finite domains, say, $I$, with simple and regular geometry may be reduced to regular Sturm–Liouville boundary value problems via separation of variables technique. This facilitates Fourier series representation of solution of such an IBVP which essentially depends on the existence of an orthonormal sequence of solutions of the underlying Sturm–Liouville boundary value problem and the linearity of the IBVP. The orthonormal sequence of solutions forms a complete orthonormal basis of $L^2(I)$. Kloosterziel [4] has made use of this idea to obtain series representation of solutions of initial value problems (IVPs) for the heat equation on infinite domains. This was achieved by constructing an orthonormal sequence of self-similar profiles of the heat equation. The series representation quickly reveals the large time behavior of the solution and is easier to compute as compared to the numerical solution. These ideas may be used to give series representation to solutions of initial value problems and initial boundary value problems for linearizable second order nonlinear PDEs. The pursuit of obtaining series representation to exact solutions of IVPs and IBVPs for linearizable nonlinear PDEs is quite interesting and fascinating, see for instance, Broadbridge [5], Pasmanter [6], Rao and Yadav [1–3]. The linearizability of general forced Burgers equation has been independently studied by many authors. Pasmanter [6] and Nimmo and Crighton [7] have used Bäcklund transformation whereas, Xu et al [8] have constructed generalized Cole-Hopf transformation for linearizing the general forced Burgers equation. Despite the linearizability of the general forced Burgers equation, the series representations to exact solutions of its IVPs and IBVPs are rare and possibly nonexistent in the literature (see [5]).

Rao and Yadav [1–3] extended the idea behind the Fourier series solution of IBVP for second order linear PDEs to initial value problems for a nonhomogeneous Burgers equation on $\mathbb{R}$. In this paper, we employ the above idea to obtain series representations of solutions of initial value problems for a damped heat equation which arises from a coupled system of forced Burgers equation on $\mathbb{R}^n$, $n \geq 1$.

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\[ u_t + uu_x = u_{xx} + \frac{kx}{(2\beta t + 1)^2}, \quad \beta \geq 0, \quad x \in \mathbb{R}, \quad t > 0, \quad (1.1) \]

subject to certain classes of bounded and unbounded initial data. In this paper, we carry out a similar study for a system of coupled nonhomogeneous Burgers equation. The solutions of Eq. (1.1) with \( \beta = 0 \) in terms of Hermite functions describe the discrete energy states of the quantum harmonic oscillator. Barndorff–Nielsen and Leonenko [10] have studied the Burgers turbulence problem with linear or quadratic external potential. The forced Burgers Eq. (1.1) corresponds to the Burgers turbulence problem with external quadratic potential. Also the solution to the forced Burgers Eq. (1.1) with \( \beta = 0 \) is directly connected to the ubiquitous Mehler’s formula of harmonic analysis and PDE (see [12]).

Let \( U : \mathbb{R}^n \times [0, \infty) \rightarrow \mathbb{R}^n, \quad n > 1 \) be the \( n \)-dimensional vector \((u_1, u_2, \ldots, u_n)\). Then, we consider an equation analogous to the nonhomogeneous Burgers Eq. (1.1) for \( U \)

\[ U_t + U \cdot \nabla U = \Delta U + \frac{kx}{(2\beta t + 1)^2}, \quad x \in \mathbb{R}^n, \quad t > 0, \quad (1.2) \]

where \( k > 0 \) and \( \beta \geq 0 \) are constants and \( x \) is the \( n \)-tuple \((x_1, x_2, \ldots, x_n)\). One may recall that Eq. (1.2) arises as a special case from the Navier–Stokes equation (see for instance, Neklyudov [13], Young et. al [14]). The term \( U \cdot \nabla U \) in (1.2) appears in the Navier–Stokes equation also and is referred as convective acceleration. The convective acceleration term is interpreted as

\[ U \cdot \nabla U = \frac{1}{2} \nabla (|U|^2) + (\nabla \times U) \times U. \quad (1.3) \]

We study Eq. (1.2) subject to the initial condition

\[ U(x, 0) = U_0(x) = (u_{01}(x), u_{02}(x), \ldots, u_{0n}(x)), \quad x \in \mathbb{R}^n. \quad (1.4) \]

The Cole–Hopf transformation (see [15])

\[ U(x, t) = -\frac{\nabla \phi(x, t)}{\phi(x, t)} \quad (1.5) \]

transforms (1.2)–(1.4) to

\[ \phi_t = \Delta \phi - \frac{k|x|^2}{4(2\beta t + 1)^2} \phi, \quad x \in \mathbb{R}^n, \quad t > 0, \quad (1.6) \]

\[ \phi(x, 0) = \Phi_0(x) = \Pi_{i=1}^n \exp \left( -\frac{1}{2} \int_{x_i}^{x_i} u_{0i} dy_i \right), \quad x \in \mathbb{R}^n; \quad (1.7) \]

here \( u_{0i} \) is as in (1.4) and \( \Pi_{i=1}^n a_i = a_1 a_2 \ldots a_n \). We refer to Joseph and Vasudev Murthy [16] for a study on the Cole–Hopf transformation for some systems of partial differential equations. We may note that in view of the transformation (1.5),

\[ \nabla \times U = 0, \quad \nabla \times U_0 = 0 \quad (1.8) \]

and (1.3) reduces to

\[ U \cdot \nabla U = \frac{1}{2} \nabla (|U|^2). \quad (1.9) \]

Here its worth pausing and make a note on the condition (1.8) which tells us that the solution \( U \) of the forced Burgers equation (1.2) is vorticity free. We refer to Bec and Khanin [9] for a wide range of applications (e.g. cosmology, vehicular traffic, directed polymers in random media) of the vorticity free solutions of forced Burgers equation in statistical physics.

Let us now define

\[ \Phi_{0i}(x) = \exp \left( -\frac{1}{2} \int_{x_i}^{x_i} u_{0i} dy_i \right), \quad x \in \mathbb{R}^n, \quad i = 1, 2, \ldots, n. \quad (1.10) \]

Then in view of (1.7), \( \Phi_0(x) = \Pi_{i=1}^n \Phi_{0i}(x) \). We deal with two classes of the initial data \( U_0 \). Section 2 deals with the initial data \( U_0 \) satisfying

\[ \Phi_{0i}(x) \in L^\infty (\mathbb{R}^n; \mathbb{R}, \mathbb{L}^2 (\mathbb{R}, e^{k|x|^2/2})), \quad i = 1, 2, \ldots, n. \quad (1.11) \]

where \( \Phi_{0i} \) is as in (1.10). The conditions in (1.11) are equivalent to

\[ \text{ess sup}_{x \in \mathbb{R}^n} \left( \int_{\mathbb{R}^n} |\Phi_{0i}(x)|^2 e^{k|x|^2/2} dx \right) < \infty. \quad (1.12) \]
If \( u_0(x) \equiv u_0(x_i) \) for all \( i = 1, 2, \ldots, n \) then the condition (1.12) is equivalent to
\[
\Phi_0(x) \in L^2(\mathbb{R}^n, e^{\beta|x|^2/2}), \quad \beta > 0,
\]  
(1.13)
where \( \Phi_0 \) in terms of \( U_0 \) is as in (1.7). We may note that the initial data \( U_0 \) satisfying (1.12) is unbounded. In Section 2, we also discuss the solution of (1.2) with initial data \( U_0 \) satisfying
\[
\Phi_0(x) \in L^\infty_{\text{loc}}(\mathbb{R}^{n-1}, L^2(\mathbb{R}, e^{-A|x|^2/2})), \quad A_1 = 4A - \beta_i, \quad i = 1, 2, \ldots, n, 
\]  
(1.14)
where \( 4A = \sqrt{\beta^2 + k - \beta} \). Here the constant \( \beta_1 \) is chosen such that \( A_1 > 0 \), that is, \( 0 < \beta_1 < \sqrt{\beta^2 + k - \beta} \). The conditions in (1.14) are equivalent to
\[
\text{ess sup}_{x \in \mathbb{R}^n} \left( \int_{\mathbb{R}} |\Phi_0(x)|^2 e^{-\beta|x|^2/2} dx \right) < \infty. 
\]  
(1.15)
If \( u_0(x) \equiv u_0(x_i) \) for all \( i = 1, 2, \ldots, n \) then the condition (1.15) is equivalent to
\[
\Phi_0(x) \in L^2(\mathbb{R}^n, e^{-A_1|x|^2/2}).
\]  
(1.16)
The class of initial data \( U_0 \) satisfying (1.15) includes bounded and compactly supported functions. It may be noted that the class of functions \( U_0 \) satisfying (1.16) contains the class of functions \( U_0 \) satisfying (1.13).

Rao and Yadav [13] obtained explicit solution of the initial value problem (IVP) (1.6) and (1.7) with \( n = 1 \). They considered two classes of initial data \( L^1(\mathbb{R}, e^{A|x|^2/2}), \beta > 0 \) and \( L^1(\mathbb{R}, e^{-A_1|x|^2/2}) \), where \( A_1 \) is as in (1.14).

With \( \Phi_0(x) \in L^2(\mathbb{R}^n, e^{A|x|^2/2}) \), the solution of the 1-d problem (1.6) and (1.7) may be written as
\[
\phi(x, t) = \sum_{n=0}^{\infty} c_n \phi_n(x, t), \quad c_n = \int_{-\infty}^{\infty} \Phi_0(x) \exp(\beta|x|^2/4) \psi_n(\sqrt{\beta}x) dx.
\]  
(1.17)
Here \( \{\phi_n(x, t)\}_{n=0}^{\infty} \) is a family of self-similar solutions of the one dimensional Eq. (1.6) and is given by
\[
\phi_n(x, t) = b^{-n}\mu(t) \exp(-\beta t^2/4) \psi_n(\sqrt{\beta^2/2}), \quad \eta = \frac{x}{b(t)}, \quad n = 0, 1, 2, \ldots,
\]  
(1.18)
where \( b(t) = \sqrt{2\beta t + 1}, \mu = \frac{1}{2b^2 + k}, \alpha_n = (2n + 1)\mu + \frac{\eta}{2} \), \( n = 0, 1, 2, \ldots \), and
\[
\psi_n(\sqrt{\beta^2/2}) = \frac{1}{\sqrt{k_n}} e^{-\mu t^2/2} H_n(\sqrt{\beta^2/2}), \quad n = 0, 1, 2, \ldots 
\]  
(1.19)
Here \( k_n^2 = 2^n n! \sqrt{\pi/\mu} \). \( H_n \) is the Hermite polynomial of order \( n \) and the Hermite polynomials are defined by the Rodrigues’ formula
\[
H_n(x) = (-1)^n e^{2x^2} \frac{d^n}{dx^n} e^{-x^2}, \quad n = 0, 1, 2, \ldots.
\]
The functions \( \psi_n(\sqrt{\beta^2/2}) \), \( n = 0, 1, 2, \ldots \), satisfy an orthonormality relation
\[
\int_{-\infty}^{\infty} \psi_n(\sqrt{\beta^2/2}) \psi_m(\sqrt{\beta^2/2}) d\eta = \delta_{nm}, \quad n, m = 0, 1, 2, \ldots,
\]
where \( \delta_{nm} \) is the Kronecker delta. Further the family of functions \( \{\psi_n(\sqrt{\beta^2/2})\}_{n=0}^{\infty} \) forms an orthonormal basis for \( L^2(\mathbb{R}) \), see Higgins [17].

With \( \Phi_0(x) \in L^2(\mathbb{R}, e^{-A_1|x|^2/2}) \), the solution of the one dimensional (1-d) problem (1.6) and (1.7) is obtained via the transformation
\[
\phi(x, t) = \exp \left( A \frac{b^{-2}(t)}{\beta} \right) \frac{b^{2A}\beta(t)}{\beta} \nu(z, \tau),
\]
\[
z = x b^{2A}/\beta(t), \quad \tau = \frac{b^{1/\beta}(t) - \lambda}{\beta}, \quad b(t) = \sqrt{2\beta t + 1},
\]
\[
\lambda = \frac{1}{4(4A + \beta)} \left( \frac{\beta}{4A + \beta} - \frac{\beta^2}{4A + \beta} \right).
\]  
(1.20)
The transformation (1.20) reduces the 1-d problem (1.6) and (1.7) to a Cauchy problem for the heat equation:
\[
\nu(x, t) = \nu_{xx}, \quad x \in \mathbb{R}, \quad t > 0,
\]
\[
\nu(x, 0) = \exp \left( -A z^2 \right) \Phi_0(z), \quad z = x.
\]  
(1.21)
(1.22)
Since \( \Phi_0 \in L^2(\mathbb{R}, e^{-A_1|x|^2/2}) \), it is easy to see that \( \nu(., 0) \in L^2(\mathbb{R}, e^{\beta|x|^2/2}) \). Here \( \beta_1 \) is chosen in such a way that
\[
0 < \beta_1 < \sqrt{\beta^2 + k - \beta}.
\]  
(1.23)
It is easy to see that sufficiently small \( \beta_i \) satisfies the requirements set above. Then following Kloosterziel \([4]\), we may write \( v(z, \tau) \) as

\[
v(z, \tau) = \sum_{n=0}^{\infty} a_n (2\beta_1 \tau + 1)^{-(n+1)/2} e^{-\beta_i \tau^2/(2\beta_1 \tau + 1)} \psi_n \left( \frac{\sqrt{\beta_i} z}{\sqrt{2(2\beta_1 \tau + 1)}} \right) = \sum_{n=0}^{\infty} a_n v_n(z, \tau),
\]

(1.24)

where \( \psi_n \) is as defined in (1.19) and

\[
a_n = \int_{\mathbb{R}} v(z, 0) e^{\beta_i \tau^2/4} \psi_n \left( \frac{\sqrt{\beta_i} z}{\sqrt{2}} \right) dz, \quad n = 0, 1, 2, \ldots
\]

(1.25)

(see [3]).

The organization of this article is as follows. In Section 2, we explicitly solve the Cauchy problem (1.6) and (1.7) with \( \Phi_0 \) satisfying (1.12) or (1.15). Section 3 considers more general classes of initial data \( \Phi_0 \) satisfying (1.13) or (1.16). Finally, Section 4 presents the conclusion of this work.

2. Solution of (1.2) subject to certain classes of bounded and unbounded initial data

In this section, we follow Kloosterziel \([4]\) and Rao and Yadav \([1,3]\) to explicitly obtain the solution of an initial value problem for (1.6) with the initial data \( \Phi_0 \) satisfying (1.12) or (1.15). One may observe that the solution of the initial value problem (1.6) and (1.7) takes the form

\[
\phi(x, t) = \Pi_{i=1}^{n} \phi_i(x, t), \quad x \in \mathbb{R}^n, \quad t \geq 0.
\]

(2.1)

We actually exploit the solution structure (2.1) for attaining the objective of this section. Here \( \phi_i(x, t), \ i = 1, 2, \ldots, n \) satisfy an initial value problem for each \( i \) of the form (1.6) and (1.7), that is,

\[
\phi_i(x, t) = \phi_i(x, 0) + k x_i \frac{x_i^2}{4(2\beta_1 t + 1)} \phi_i(x, 0), \quad x \in \mathbb{R}^n, \quad t > 0,
\]

(2.2)

\[
\phi_i(x, 0) = \Phi_0(x), \quad x \in \mathbb{R}^n.
\]

(2.3)

Thus for \( \Phi_0 \) satisfying (1.12) and \( \phi_i(x, t) \) solving the IVP (2.2) and (2.3), we may, in view of (2.1), expand \( \phi_i(x, 0) \) as

\[
\phi_i(x, 0) = \sum_{m=0}^{\infty} c_{m}^{(i)}(x) \phi_m(x, 0), \quad \partial_{x_i} c_{m}^{(i)}(x) = 0, \ x \in \mathbb{R}^{n-1}, \quad i = 1, 2, \ldots, n.
\]

(2.4)

The sequence \( \{ \phi_m(x_0, 0) \} \) appearing in (2.4) forms an orthonormal basis of \( L^2(\mathbb{R}, e^{\beta_i \tau^2/2}) \) and is obtained from the family of self-similar solutions \( \{ \phi_m(x, t) \} \) of (2.2) (see (1.18)). Then \( \phi_i(x, t) \) is given by

\[
\phi_i(x, t) = \sum_{m=0}^{\infty} c_{m}^{(i)}(x) \phi_m(x, 0), \quad i = 1, 2, \ldots, n
\]

(2.5)

(see (1.17)). From the convergence of the series (1.17) in \( L^2(\mathbb{R} \times [0, T], e^{\beta_i \tau^2/2}) \) for any \( T > 0 \) to the solution \( \phi(x, t) \) of the 1-d problem (1.6) and (1.7) (see Rao and Yadav [1]), it is easy to see that the series (2.5) converges in \( L^\infty(\mathbb{R}^{n-1}, L^2(\mathbb{R} \times [0, T], e^{\beta_i \tau^2/2}) \), \( \eta_i = x_i/b(t) \) for each \( i \) and for any \( T > 0 \). We may note that if \( u_{0i} \equiv u_{0i}(x), \ i = 1, 2, \ldots, n \) then \( \phi(x, t) \) given by (2.1) converges in \( L^2(\mathbb{R}^n \times [0, T], e^{\beta_i \tau^2/2}) \), \( \eta^2 = \sum_{i=1}^{n} \eta_i^2, \ T > 0 \).

When the initial data \( \Phi_0 \) satisfies (1.15) then \( \phi_i(x, t), \ i = 1, 2, \ldots, n \) is given by (1.24) via the transformation (1.20). That is,

\[
\phi_i(x, t) = \exp \left( A x_i^2 b^{-2}(t) \right) b^{2A/\beta}(t) v_i(z, \tau), \quad z = x b^{4A/\beta}(t), \ x \in \mathbb{R}^n,
\]

(2.6)

where \( A \) and \( \tau \) are as in (1.20) and \( v_i(z, \tau) \) satisfies

\[
v_i(z, \tau) = v_i(z_0), \quad z_i = x_i b^{4A/\beta}(t), \quad \tau > 0,
\]

(2.7)

\[
v_i(z_0) = \exp \left( -A z_i^2 \right) \Phi_0(z), \quad z = x.
\]

(2.8)

Since \( \Phi_0(z), \ i = 1, 2, \ldots, n \) satisfy (1.14) and \( v_i(z, \tau) \) solves the IVP (2.7) and (2.8), we may expand \( v_i(z, 0) \) as

\[
v_i(z, 0) = \sum_{m=0}^{\infty} a_m^{(i)}(z) v_m(z, 0), \quad \partial_{z_i} a_m^{(i)}(z) = 0, \quad z \in \mathbb{R}^{n-1}.
\]

(2.9)

The sequence \( \{ v_m(z, 0) \} \) appearing in (2.9) forms an orthonormal basis of \( L^2(\mathbb{R}, e^{\beta_i \tau^2/2}) \) and is obtained from the family of self-similar solutions \( \{ v_m(z, t) \} \) of the heat Eq. (2.7). Mimicking the series (1.24) we may write \( v_i(z, \tau) \) as
\( v^{(i)}(z, \tau) = \sum_{m=0}^{\infty} a_m^{(i)}(z) \psi_m(z, \tau), \quad i = 1, 2, \ldots, n, \)  

(2.10)

where \( \psi_m(z, \tau) \) is obtained from (1.24). From the convergence of \( \phi(x, t) \) given by (1.20) in \( L^2(\mathbb{R} \times [0, T], e^{\beta_1 z^2/2}) \), where \( \beta_1 \) is as in (1.23), \( z = z/\sqrt{2\beta_1 \tau + 1} \) and \( T > 0 \) (see [3]), it is easy to see that the series

\[
\phi^{(i)}(x, t) = b^{2A_2}(t) \sum_{m=0}^{\infty} a_m^{(i)}(x)e^{A_2 k^2 - 4 \tilde{z}_i} \psi_m(z, \tau), \quad i = 1, 2, \ldots, n
\]

(2.11)

converges in \( L^\infty(\mathbb{R}^{n-1}, L^2(\mathbb{R} \times [0, T], e^{\tilde{z}_i z^2/2})) \), \( \tilde{z}_i = z/\sqrt{2\beta_1 \tau + 1} \) for each \( i \) and for any \( T > 0 \). We may note that if \( u_0 = u_0(x) \), \( i = 1, 2, \ldots, n \) then \( \phi(x, t) \) given by (2.1) converges in \( L^2(\mathbb{R}^{n-1}, e^{\tilde{z}_i z^2/2}) \), \( z^2 = \sum_{i=1}^{n} z_i^2 \), \( T > 0 \).

3. Solutions of (1.6) in terms of \( n \)-dimensional Hermite functions

In this section, we obtain the solution of the Cauchy problem (1.6) and (1.7) in terms of \( n \)-dimensional scaled Hermite functions. We consider the initial data \( u_0(x) \) such that \( \Phi_0(x) \) satisfies (1.13) or (1.16) irrespective of the fact that \( u_0 = u_0(x) \), \( i = 1, 2, \ldots, n \).

For this purpose, we consider the solution of (1.6) in the form

\[
\phi(x, t) = \frac{1}{a(t)} f(\eta), \quad \eta = \frac{x}{b(t)}, \quad x \in \mathbb{R}^n, \quad t > 0;
\]

(3.1)

here \( a(t) \) and \( b(t) \) are such that \( a(0) = b(0) = 1 \). Now substituting (3.1) in (1.6), we get

\[
\Delta f + b(t)b'(t) \eta \cdot \nabla f + \frac{b^2(t) a'(t)}{a(t)} f(\eta) - \frac{kb^4(t)}{4(2\beta t + 1)^2} |\eta|^2 f(\eta) = 0.
\]

(3.2)

The functions \( a(t) \) and \( b(t) \) will be determined in such a way that (3.2) is free of the time dependent coefficients, that is,

\[
b(t)b'(t) = \tilde{b}_1, \quad \frac{b^2(t) a'(t)}{a(t)} = \alpha, \quad b^4(t) = (2\beta t + 1)^2.
\]

(3.3)

Here \( \alpha \) and \( \tilde{b}_1 \) are some constants to be determined. Integrating the first two relations in (3.3) subject to \( a(0) = b(0) = 1 \) and making use of the third relation in (3.3), we arrive at \( \tilde{b}_1 = \beta \) and

\[
a(t) = b^{2/\beta}(t), \quad b(t) = \sqrt{2\beta t + 1}, \quad \text{if} \ \beta > 0;
\]

(3.4)

\[
a(t) = e^{\pi t}, \quad b(t) = 1, \quad \text{if} \ \beta = 0.
\]

(3.5)

Here we discuss only the first case \( \beta > 0 \). The second case \( \beta = 0 \) easily follows from the first case.

Case 1. \( \beta > 0 \).

The transformation (3.1) together with (3.4) transforms (1.6) to a second order linear elliptic PDE

\[
\Delta f + \beta \eta \cdot \nabla f + \left( \alpha - \frac{k}{4} |\eta|^2 \right) f = 0,
\]

(3.6)

where \( \alpha \) is an arbitrary constant. Introducing the change of variable

\[
f(\eta) = e^{-\rho \eta^2/2} u(\eta),
\]

(3.7)

the PDE (3.6) transforms to

\[
\Delta u + \left( \alpha - \frac{\eta \beta}{2} \right) - \frac{1}{4} (\beta^2 + k) |\eta|^2 \nu = 0.
\]

(3.8)

Since \( \beta^2 + k > 0 \), set

\[
\mu = \frac{1}{2} \sqrt{\beta^2 + k}, \quad \alpha = \alpha + \frac{n \beta}{2}, \quad |\eta|^2 = \sum_{i=1}^{n} \eta_i^2, \quad \gamma \in \mathbb{N}^n.
\]

(3.9)

With the above choice of \( \alpha = \alpha \gamma \), we get the \( n \)-dimensional scaled Hermite functions \( \Psi^{\mu \gamma}(\eta) \) as solutions of (3.8). The \( n \)-dimensional scaled Hermite functions are given by

\[
\Psi^{\mu \gamma}(\eta) = \mu^{n/4} \Psi^{\mu}(\mu^{1/2} \eta), \quad \eta \in \mathbb{N}^n,
\]

(3.10)

where \( \Psi^{\mu}(\eta) \) are the \( n \)-dimensional Hermite functions defined as

\[
\Psi^{\mu}(\eta) = \Pi_{i=1}^{n} \psi^{\mu \gamma}(\eta_i), \quad \eta_i = \frac{x_i}{b(t)} , \quad \eta \in \mathbb{N}^n.
\]

(3.11)
Here $\psi_n(\eta)$ are the one dimensional Hermite functions.  

Thus, we obtain a family of solutions of (1.6) given by

$$\phi^\eta_j(x, t) = b^{-\beta/2}(t) e^{-|\eta|^2/4} \psi_\eta^\eta(\eta), \quad \eta \in \mathbb{H}^n. \quad (3.12)$$

We may note that the family $\{\phi^\eta_j(x, 0) : \eta \in \mathbb{H}^n\}$ forms an orthonormal basis of $L^2(\mathbb{H}^n, e^{\beta |x|^2/2})$. In this section, we have assumed that $\Phi_0(x) \in L^2(\mathbb{R}^n, e^{\beta |x|^2/2})$, therefore,

$$\Phi_0(x) = \sum_{\eta \in \mathbb{H}^n} c_\eta \phi^\eta_j(x, 0), \quad c_\eta = \int_{\mathbb{H}^n} \Phi_0(x) \phi^\eta_j(x, 0) e^{\beta |x|^2/2} dx. \quad (3.13)$$

By the superposition principle, the solution of the Cauchy problem (1.6) and (1.7) may be written as

$$\phi(x, t) = \sum_{\eta \in \mathbb{H}^n} c_\eta \phi^\eta_j(x, t). \quad (3.14)$$

Since the solution of the Cauchy problem (1.2)–(1.4) is to be obtained in terms of $\phi(x, t)$ given by (3.14) and $\nabla \phi(x, t)$ via the Cole–Hopf transformation (1.5), we formally differentiate term-by-term the series (3.14) and write

$$\phi_n(x, t) = \sum_{\eta \in \mathbb{H}^n} c_\eta \partial_x \phi^\eta_j(x, t), \quad i = 1, 2, \ldots, n. \quad (3.15)$$

Now, we discuss the convergence of the series (3.14) and (3.15). Let us define

$$S_m \phi(x, t) = \sum_{|| \eta || \leq m} c_\eta \phi^\eta_j(x, t), \quad m \in \mathbb{N}, \quad (3.16)$$

$$S_m \phi(x, t) = \sum_{|| \eta || \leq m} c_\eta \phi^\eta_j(x, t), \quad i = 1, 2, \ldots, n. \quad (3.17)$$

Now we present some convergence results for the partial sums $S_m \phi$ and $(S_m \phi)_i$. Before that we have the following lemma.

**Lemma 3.1.** Let $f \in L^2(\mathbb{R}, e^{\alpha \beta^2/2}), \beta > 0$ and

$$S_m f(x) = \sum_{n=0}^m e^{-\alpha \beta^2/4} \psi_n(x) (f, e^{-\alpha \beta^2/4} \psi_n)_{L^2(\mathbb{R}, \mathbb{R})}, \quad m \in \mathbb{N}. \quad (3.18)$$

Then

(i) $S_m f \to f$ in $L^2(\mathbb{R}, e^{\alpha \beta^2/2})$ as $m \to \infty$.

(ii) $(S_m f)' + \beta \xi S_m f \to f'$ weakly in $H^{-1}(\mathbb{R}, e^{\alpha \beta^2/2})$ as $m \to \infty$.

**Proof.** The proof of (i) is straightforward, since $\{e^{-\alpha \beta^2/4} \psi_n(x) : n \in \mathbb{N}\}$ forms an orthonormal basis of $L^2(\mathbb{R}, e^{\alpha \beta^2/2})$.

For the proof of (ii), we may note that

$$(S_m f)' + \beta \xi S_m f = e^{-\alpha \beta^2/2} (e^{\alpha \beta^2/2} S_m f)' \quad (3.19)$$

Thus,

$$((S_m f)' + \beta \xi S_m f)' g_{H^{-1}(\mathbb{R}, e^{\alpha \beta^2/2})} = \int_{\mathbb{R}} (e^{-\alpha \beta^2/2} (S_m f - f))' (x) g(x) dx$$

$$= \int_{\mathbb{R}} (f - S_m f)(x) g'(x) e^{\alpha \beta^2/2} dx = (f - S_m f, g')_{L^2(\mathbb{R}, e^{\alpha \beta^2/2})} \quad (3.18)$$

for all $g \in H^1(\mathbb{R}, e^{\alpha \beta^2/2})$. Then

$$||((S_m f)' + \beta \xi S_m f)' g||_{H^{-1}(\mathbb{R}, e^{\alpha \beta^2/2})} \leq ||S_m f - f||_{L^2(\mathbb{R}, e^{\alpha \beta^2/2})} ||g'||_{L^2(\mathbb{R}, e^{\alpha \beta^2/2})} \leq ||S_m f - f||_{L^2(\mathbb{R}, e^{\alpha \beta^2/2})} ||g||_{H^1(\mathbb{R}, e^{\alpha \beta^2/2})}.$$

From (3.19) and Lemma 3.1 (i), it is easy to conclude (ii). □

Now we have the following proposition.

**Proposition 3.2.** Let $\mathbb{R}^\beta_T := \mathbb{R}^n \times [0, T]$ for any $T > 0$. Then

(i) $S_m \phi \to \phi$ in $L^2(\mathbb{R}^\beta_T, e^{\alpha \beta^2/2})$ as $m \to \infty$.

(ii) $(S_m \phi)_i + \beta \xi S_m \phi \to \phi_i$ weakly in $H^{-1}(\mathbb{R}^\beta_T, e^{\alpha \beta^2/2})$ as $m \to \infty$ for $i = 1, 2, \ldots, n$. 

Proof. Proof of (i) follows from Rao and Yadav [1] (see Appendix). Proof of (ii) follows from (i) and Lemma 3.1 (ii). □

We now consider the initial data $U_0(x)$ such that $\Phi_0(x)$ satisfies (1.16), that is, $\Phi_0(x) \in L^2(\mathbb{R}^n, e^{-|x|^2/2})$. To deal with such initial data, we use the transformation

$$
\phi(x, t) = \exp \left( Ax^2 b(t) \right) b^{2n/b}(t) \psi(z, \tau),
$$

$$
z = x^2 b^{1/2}(t), \quad \tau = 2\beta t + 1, \quad b(t) = \sqrt{2\beta t + 1},
$$

$$
\lambda = \frac{1}{2(4A + \beta)}, \quad A = \frac{1}{4}(-\beta + \sqrt{\beta^2 + 4k})
$$

(see (1.20)) to transform the initial value problem (1.6) and (1.7) to a Cauchy problem for the heat equation:

$$
v_t = \Delta v, \quad z \in \mathbb{R}^n, \quad \tau > 0,
$$

$$
v(z, 0) = \exp \left(-Az^2\right) \Phi_0(z), \quad z = x.
$$

(3.19)

In view of (3.21), the condition (1.16) on $\Phi_0(x)$ leads to the condition $v(z, 0) \in L^2(\mathbb{R}^n, e^{\alpha |x|^2/2})$, where $\beta_1$ is a constant chosen as in (1.23). Thus the solution of the Cauchy problem (3.20) and (3.21) may be written in terms of the $n$-dimensional self-similar solutions

$$
v^{\mu}_n(z, \tau) = e^{-\beta_1 |z|^2/(8|\beta|^{1/2} - 1)} \psi^{\mu}_n(z/\sqrt{2\beta_1}^n + 1), \quad \nu = \beta_1 \tau, \quad \gamma \in \mathbb{N}^n
$$

(3.22)

of the heat Eq. (3.20). Here $\psi^{\mu}_n$ are the scaled $n$-dimensional Hermite functions as defined in (3.10) and (3.11). The solution of the Cauchy problem (3.20) and (3.21) may be written as

$$
v(z, \tau) = \sum_{\gamma \in \mathbb{N}^n} A^{(2\beta_1 \tau + 1)}_{\gamma} e^{b_1 |\gamma|^2/2} v^{\mu}_n(z, \tau),
$$

$$
a^{(\gamma)} = \int_{\mathbb{R}^n} v(z, 0) v^{\mu}_n(z, 0) e^{b_1 |\gamma|^2/2} dz.
$$

(3.23)

where $\gamma$ is a multi-index and $|\gamma| = \sum_{i=1}^n \gamma_i$. Now the solution of the initial value problem (1.6) and (1.7) subject to the condition (1.16) is obtained via the transformation (3.19) and the series (3.23).

4. Conclusions

This article is about obtaining explicit solutions of Cauchy problems for the system of forced Burgers Eq. (1.2) on $\mathbb{R}^n \times (0, \infty)$. The Cole-Hopf transformation (1.5) linearizes (1.2) to the linear PDE (1.6). Further, the linear PDE (1.6) can also be transformed to the heat Eq. (3.20) via the transformation (3.19). We have considered Cauchy problems for (1.6) as well as the heat Eq. (3.20) subject to various classes of initial data.

Section 2 exploits the solution structure (2.1) of the linear PDE (1.6). The $n$-dimensional heat equation also has the same solution structure. The solutions of Cauchy problems for (1.6) and (1.7) with the initial data $\Phi_0$ satisfying (1.12) or (1.15) are obtained as products of $n$ copies of the one dimensional series solution (2.5) or (2.6).

Finally, Section 3 deals with Cauchy problems for (1.6) with initial data $\Phi_0$ satisfying (1.13) or (1.16). Here the solutions are expressed in terms of a family of $n$-dimensional scaled Hermite functions.

References