COMPLEXITY OF CERTAIN FUNCTIONAL VARIANTS OF TOTAL DOMINATION IN CHORDAL BIPARTITE GRAPHS

D. PRADHAN

Department of Computer Science and Automation
Indian Institute of Science, Bangalore-560012, India
dina@csa.iisc.ernet.in

Received 3 February 2012
Revised 13 May 2012
Published 1 August 2012

In this paper, we consider minimum total domination problem along with two of its variations namely, minimum signed total domination problem and minimum minus total domination problem for chordal bipartite graphs. In the minimum total domination problem, the objective is to find a smallest size subset \( TD \subseteq V \) of a given graph \( G = (V, E) \) such that \( |TD \cap N_G(v)| \geq 1 \) for every \( v \in V \). In the minimum signed (minus) total domination problem for a graph \( G = (V, E) \), it is required to find a function \( f : V \rightarrow \{-1, 0, 1\} \) such that \( f(N_G(v)) = \sum_{u \in N_G(v)} f(u) \geq 1 \) for each \( v \in V \), and the cost \( f(V) = \sum_{v \in V} f(v) \) is minimized. We first show that for a given chordal bipartite graph \( G = (V, E) \) with a weak elimination ordering, a minimum total dominating set can be computed in \( O(n + m) \) time, where \( n = |V| \) and \( m = |E| \). This improves the complexity of the minimum total domination problem for chordal bipartite graphs from \( O(n^2) \) time to \( O(n + m) \) time. We then adopt a unified approach to solve the minimum signed (minus) total domination problem for chordal bipartite graphs in \( O(n + m) \) time. The method is also able to solve the minimum \( k \)-tuple total domination problem for chordal bipartite graphs in \( O(n + m) \) time. For a fixed integer \( k \geq 1 \) and a graph \( G = (V, E) \), the minimum \( k \)-tuple total domination problem is to find a smallest subset \( TD_k \subseteq V \) such that \( |TD_k \cap N_G(v)| \geq k \) for every \( v \in V \).

Keywords: Domination; total domination; \( k \)-tuple total domination; minus total domination; signed total domination; NP-complete.

Mathematics Subject Classifications 2010: 05C69, 05C85, 68R10

1. Introduction

For a graph \( G = (V, E) \), the sets \( N_G(v) = \{ u \in V | uv \in E \} \) and \( N_G[v] = N_G(v) \cup \{v\} \) denote the neighborhood and the closed neighborhood of a vertex \( v \), respectively. A set \( D \) of vertices of a graph \( G = (V, E) \) is a dominating set of \( G \) if every vertex in
Let \( D \) be a subset of \( V \). A subset \( D \) of \( V \) is a dominating set of \( G \) if \( |N_G(v) \cap D| \geq 1 \) for every \( v \in V \). The domination number of a graph \( G \), denoted by \( \gamma(G) \), is the minimum cardinality of a dominating set of \( G \). It is widely studied in literature (see [9, 10]).

Among the variations of domination, total domination is one of those. A set \( D \) of vertices of a graph \( G = (V, E) \) is a total dominating set of \( G \) if every vertex in \( V \) is adjacent to at least one vertex of \( D \). Equivalently, a subset \( D \) of \( V \) is a total dominating set of \( G \) if \( |N_G(v) \cap D| \geq 1 \) for every \( v \in V \). The total domination number of a graph \( G \), denoted by \( \gamma_t(G) \), is the minimum cardinality of a total dominating set of \( G \). For a given graph \( G = (V, E) \) and a positive integer \( l \), the Min-Total-Dom-Set problem is to decide whether \( G \) has a total dominating set of size at most \( l \). For extensive literature and survey of total domination in graphs, we refer to [9–11]. As a variation of domination, the concept of \( k \)-tuple total domination has been introduced by Henning and Kazemi [13]. For a fixed positive integer \( k \), a \( k \)-tuple total dominating set of a graph \( G = (V, E) \) is a subset \( TD_k \) of \( V \) such that every vertex in \( V \) is adjacent to at least \( k \) vertices of \( TD_k \). Equivalently, a set \( TD_k \subseteq V \) is a \( k \)-tuple total dominating set if \( |N_G(v) \cap TD_k| \geq k \) for every \( v \in V \). So \( k \)-tuple total domination is the generalization of the usual total domination. The \( k \)-tuple total domination number of a graph \( G \), denoted by \( \gamma_{k,t}(G) \), is the minimum cardinality of a \( k \)-tuple total dominating set of \( G \). For a given graph \( G = (V, E) \) and a positive integer \( l \), the Min-\( k \)-Tuple-Total-Dom-Set problem is to decide whether \( G \) has a \( k \)-tuple total dominating set of size at most \( l \).

Let \( \mathcal{P} \) be a subset of integers. A \( \mathcal{P} \)-valued function of a graph \( G = (V, E) \) is a function \( f: V \rightarrow \mathcal{P} \) which assigns a value in \( \mathcal{P} \) to each vertex \( v \in V \). The set \( \mathcal{P} \) is called the weight set of \( f \). For any set \( S \subseteq V \), \( f(S) = \sum_{v \in S} f(v) \) is called the weight of the set \( S \). The function \( f \) is called a \( \mathcal{P} \)-total dominating function of \( G = (V, E) \) if \( f(N_G(v)) \geq 1 \) for every \( v \in V \). A total dominating set can be viewed as a \( \mathcal{P} \)-total dominating function \( f \) with \( \mathcal{P} = \{0,1\} \). The weight function is the weight of a total dominating function and \( \gamma(f) = \min\{f(V) | f \) is a total dominating function\}. A \( \mathcal{P} \)-total dominating function is called a signed (resp. minus) total dominating function if \( \mathcal{P} = \{-1,1\} \) (resp. \( \{-1,0,1\} \)). The signed (resp. minus) total domination number, denoted by \( \gamma_{s,t}(G) \) (resp. \( \gamma_{t,-}(G) \)), is the minimum weight of a signed (resp. minus) total dominating function of \( G \). For a given graph \( G = (V, E) \) and a positive integer \( l \), the Min-Signed-Total-Dom-Function (resp. Min-Minus-Total-Dom-Function) problem is to decide whether \( G \) has a signed (resp. minus) total dominating function of \( G \) of size at most \( l \). The literature on signed (minus) total domination function can be found in [12, 14, 15, 18, 19, 22–24].

A bipartite graph \( G = (X, Y, E) \) is called chordal bipartite if any induced cycle of length at least six has a chord i.e., an edge between two nonconsecutive vertices of the cycle. Chordal bipartite graphs are bipartite analog of chordal
graphs and are characterized in terms of weak elimination ordering [21] and strong $T$-elimination ordering [3]. Using weak elimination ordering of a chordal bipartite graph researchers have designed efficient algorithms for many graph problems (see [21]). A natural question arises whether it is possible to design efficient algorithms for finding minimum total dominating set and minimum signed (minus) total dominating set for a given chordal bipartite graph using the weak elimination ordering associated with it. We answer to this question positively in this paper. In particular, we first propose an $O(n + m)$ time algorithm for computing a minimum total dominating set in a chordal bipartite graph $G$ if a weak elimination ordering of $G$ is provided. This improves the complexity of finding a minimum total dominating set in chordal bipartite graph from $O(n^2)$ time (previously known [4]) to $O(n + m)$ time. Then we adopt a unified method and show that the signed (minus) total domination problem can be solved in $O(n + m)$ time for a chordal bipartite graph $G$ if a weak elimination ordering of $G$ is provided. This improves the complexity of finding the minimum weight of signed (minus) total domination function for chordal bipartite graphs proposed in [15] from $O(n^2)$ time to $O(n + m)$ time.

On the negative side, we show that Min-Total-Dom-Set problem is NP-complete for perfect edge elimination bipartite graphs. Similarly, we show that Min-Signed-Total-Dom-Function problem is NP-complete for perfect edge elimination bipartite graphs and for graphs with a $b$-extremal ordering. We again show that Min-Signed-Total-Dom-Function problem is NP-complete for perfect edge elimination bipartite graphs. Lastly, we show that Min-$k$-Tuple-Total-Dom-Set problem is NP-complete for perfect edge elimination bipartite graphs and also for graphs with a $b$-extremal ordering.

2. Preliminaries

For a graph $G = (V, E)$, the degree of a vertex $v$ is $|N_G(v)|$ and is denoted by $d_G(v)$. Let $n$ and $m$ denote the number of vertices and number of edges of a graph $G$, respectively. For $S \subseteq V$, let $G[S]$ denote the subgraph induced by $G$ on $S$. If $G[C]$, $C \subseteq V$, is a maximal complete subgraph of $G$, then $C$ is called a clique of $G$. A set $S \subseteq V$ is an independent set if $G[S]$ has no edge. For a set $S \subseteq V$, we denote $N(S) = \cup_{v \in S} N_G(v)$.

A graph is said to be a chordal graph if every cycle of length at least four has a chord. A chordal graph $G = (V, E)$ is a split graph if $V$ can be partitioned into two sets $K$ and $S$ such that $K$ is a clique and $S$ is an independent set. A vertex $v \in V(G)$ is a simplicial vertex of $G$ if $N_G[v]$ is a clique of $G$. An ordering $\alpha = (v_1, v_2, \ldots, v_n)$ is a perfect elimination ordering (PEO) of $G$ if $v_i$ is a simplicial vertex of $G_i = G[\{v_i, v_{i+1}, \ldots, v_n\}] \forall i, 1 \leq i \leq n$. It is characterized that a graph $G$ is chordal if and only if it has a PEO [6]. A perfect elimination ordering $(v_1, v_2, \ldots, v_n)$ is called a strong elimination ordering if for any $i \leq j \leq k$, $v_j, v_k \in N_G[v_i]$, then
It is known that a graph is strongly chordal if and only if it admits a strong elimination ordering \([5]\). Currently, the fastest algorithms for recognizing strongly chordal graphs and computing a strong elimination ordering take \(O(m \log n)\) \([16]\) or \(O(n^2)\) time \([20]\).

A graph \(G = (V,E)\) is called a bipartite graph if \(V(G)\) can be partitioned into two disjoint sets \(X\) and \(Y\) such that every edge of \(G\) joins a vertex in \(X\) to another vertex in \(Y\). A partition \((X,Y)\) of \(V\) is called a bipartition. A bipartite graph with bipartition \((X,Y)\) of \(V\) is denoted by \(G = (X,Y,E)\). For a bipartite graph \(G = (X,Y,E)\), we denote \(n_x\) and \(n_y\) as the cardinalities of the sets \(X\) and \(Y\), respectively. A bipartite graph \(G = (X,Y,E)\) is a complete bipartite graph if every vertex \(x\) of \(X\) is adjacent to every vertex \(y\) of \(Y\). A complete bipartite graph \(G = (X,Y,E)\) with \(|X| = r\) and \(|Y| = s\) is denoted as \(K_{r,s}\).

A vertex \(v\) in \(V\) is called a weak simplicial vertex of a graph \(G = (V,E)\) if \(N_G(v)\) is an independent set and for each \(x,y \in N_G(v)\) either \(N_G(x) \subseteq N_G(y)\) or \(N_G(y) \subseteq N_G(x)\). An ordering \(\sigma = (v_1, v_2, \ldots, v_n)\) of \(V\) of a graph \(G = (V,E)\) is called a weak elimination ordering of \(G\) if each \(v_i\) is weak simplicial in \(G_i = G[\{v_i, v_{i+1}, \ldots, v_n\}]\) and for each \(v_j, v_k \in N_G(v_i)\) with \(j < k\), \(N_G(v_j) \subseteq N_G(v_k)\).

Let \(G = (X,Y,E)\) be a bipartite graph and \(\alpha = (x_1, x_2, \ldots, x_{n_x})\) and \(\beta = (y_1, y_2, \ldots, y_{n_y})\) be some orderings of \(X\) and \(Y\), respectively. The ordering of vertices, \(\alpha = (x_1, x_2, \ldots, x_{n_x})\) and \(\beta = (y_1, y_2, \ldots, y_{n_y})\) is a strong \(T\)-elimination ordering if for each \(1 \leq i \leq n_y\) and \(1 \leq j < k \leq n_x\) where \(x_j, x_k \in N_G(y_i)\), we have that \(N_G'(x_j) \subseteq N_G'(x_k)\), where \(G' = G[\{y_i, y_{i+1}, \ldots, y_{n_y}\} \cup \{x_1, x_2, \ldots, x_{n_x}\}]\).

A bipartite graph \(G = (X,Y,E)\) is a chordal bipartite graph if every cycle of \(G\) of length at least six has a chord, i.e., an edge joining two nonconsecutive vertices of the cycle. Chordal bipartite graphs are characterized in terms of weak elimination ordering \([21]\) and are also characterized in terms of strong \(T\)-elimination ordering \([3]\). Given a chordal bipartite graph \(G = (V,E)\), an algorithm that computes a weak elimination ordering of \(G\) is given in \([21]\). It was claimed that the algorithm runs in linear time. However, as pointed out by the author himself \([21]\), the complexity analysis of the algorithm is not correct. The actual time complexity of the algorithm remains \(O(\min\{m \log n, n^2\})\). So computing a weak elimination ordering of a chordal bipartite graph \(G\) takes \(O(\min\{m \log n, n^2\})\) time. Below, we present the relation between a weak elimination ordering and a strong \(T\)-elimination ordering of a chordal bipartite graph \(G = (X,Y,E)\).

Let \(\sigma = (v_1, v_2, \ldots, v_r)\) be a weak elimination ordering of a chordal bipartite graph \(G = (X,Y,E)\). Now let \(\sigma_X = (x_1, x_2, \ldots, x_{n_x})\) (resp. \(\sigma_Y = (y_1, y_2, \ldots, y_{n_y})\)) be an ordering of \(X\) (resp. \(Y\)) such that \(i < j\) implies \(x_i\) (resp. \(y_i\)) precedes \(x_j\) (resp. \(y_j\)) in \(\sigma\).

**Lemma 2.1.** \(\sigma_X = (x_1, x_2, \ldots, x_{n_x})\) and \(\sigma_Y = (y_1, y_2, \ldots, y_{n_y})\) is a strong \(T\)-elimination ordering of \(G\).
Proof. For a vertex \( y_i \), let \( x_j, x_k \in N_{G'}(y_i) \) with \( j < k \), where \( G' = G'[\{y_i, y_{i+1}, \ldots, y_{n_y}\} \cup \{x_1, x_2, \ldots, x_{n_x}\}] \). Let \( u \in N_{G'}(x_j) \). We need to show that \( u x_k \in E(G) \). If \( i < j < k \) with respect to \( \sigma \), then it is clear that \( u x_k \in E(G) \) as \( N_{G'}(x_j) \subseteq N_{G'}(x_k) \). If \( j < i < k \) or \( j < k < i \) with respect to \( \sigma \), then \( x_k \in N_{G'}(y_i) \) for any \( y_r \in N_{G'}(x_j) \). This implies that \( u x_k \in E(G) \). Therefore, \( \sigma_X = (x_1, x_2, \ldots, x_{n_x}) \) and \( \sigma_Y = (y_1, y_2, \ldots, y_{n_y}) \) is a strong T-elimination ordering of \( G \).

Note that in Lemma 2.1, we also can argue taking \( y_j, y_k \in N_{G'}(x_i) \) for any vertex \( x_i \in X \) with \( j < k \), where \( G' = G'[\{x_i, x_{i+1}, \ldots, x_{n_x}\} \cup \{y_1, y_2, \ldots, y_{n_y}\}] \) and can show that \( N_{G'}(y_j) \subseteq N_{G'}(y_k) \). Since we scan all the vertices of \( G \) with respect to \( \sigma \), we have the following lemma.

**Lemma 2.2.** Given a weak elimination ordering \( \sigma \) of a chordal bipartite graph \( G = (X, Y, E) \), a strong T-elimination ordering \( \sigma_X = (x_1, x_2, \ldots, x_{n_x}) \) and \( \sigma_Y = (y_1, y_2, \ldots, y_{n_y}) \) of \( G \) can be obtained in \( O(n) \) time such that

- for each \( x_i \), \( 1 \leq i \leq n_x \), \( N_{G'}(y_j) \subseteq N_{G'}(y_k) \), where \( G' = G'[\{x_i, x_{i+1}, \ldots, x_{n_x}\} \cup \{y_1, y_2, \ldots, y_{n_y}\}] \) and \( y_j, y_k \in N_{G'}(x_i) \) with \( j < k \).
- for each \( y_j \), \( 1 \leq i \leq n_y \), \( N_{G'}(x_j) \subseteq N_{G'}(x_k) \), where \( G'' = G'[\{y_i, y_{i+1}, \ldots, y_{n_y}\} \cup \{x_1, x_2, \ldots, x_{n_x}\}] \) and \( x_j, x_k \in N_{G'}(y_i) \) with \( j < k \).

Let \( G = (X, Y, E) \) be a bipartite graph. An edge \( e = xy \) is said to be bisimplicial edge if \( G[N_G(x) \cup N_G(y)] \) is a complete bipartite subgraph of \( G \). Let \( \sigma = (x_1y_1, x_2y_2, \ldots, x_ky_k) \) be an ordering of pairwise nonadjacent edges of \( G \). Denote \( S_j = \{x_1, x_2, \ldots, x_j\} \cup \{y_1, y_2, \ldots, y_j\} \) and let \( S_0 = \emptyset \). The ordering \( \sigma = (x_1y_1, x_2y_2, \ldots, x_ky_k) \) is called a perfect edge elimination ordering of \( G \) if \( x_{j+1}y_{j+1} \) is a bisimplicial edge in \( G[(X \cup Y) \setminus S_j] \) for \( j = 0, 1, \ldots, k - 1 \) and \( G[(X \cup Y) \setminus S_k] \) has no edge. A graph for which there exists a perfect edge elimination ordering is called a perfect elimination bipartite graph. The class of perfect elimination bipartite graphs includes the class of chordal bipartite graphs and has been introduced by Golombic and Goss [7].

Next we give the definition of graphs with a \( b \)-extremal ordering. A vertex \( v \) in a graph \( G = (V, E) \) is \( b \)-extremal if \( N_G(v) \subseteq N_G(u) \) for some \( u \in V \) with \( u \neq v \) and there exists a vertex \( w \in V \) such that \( N(N_G(v)) = N_G(w) \). Note that \( w \) need not be different from \( u \). The ordering \( (v_1, v_2, \ldots, v_n) \) of \( V \) is a \( b \)-extremal ordering of \( G \) if \( v_i \) is \( b \)-extremal in \( G_i \) for each \( 1 \leq i \leq n \). Note that the graph that admits a \( b \)-extremal ordering must be bipartite.

A vertex \( y \in N_G(x) \) of a bipartite graph \( G = (X, Y, E) \) is called a maximum neighbor of \( x \) if for all \( y' \in N_G(x) \), \( N_G(y') \subseteq N_G(y) \) holds. Let \( G^y_x = G[X \cup \{y, y_{i+1}, \ldots, y_{n_y}\}] \). A linear ordering \( (y_1, y_2, \ldots, y_{n_y}) \) of \( Y \) called a
maximum X-neighborhood ordering of $G$ if for all $i \in \{1, 2, \ldots, n\}$, there is a maximum neighbor $x_i \in N_G(y_i)$ of $y_i$ in $G^Y$ i.e., $N^Y_G(x) \subseteq N^Y_G(x_i)$ holds for all $x \in N_G(y_i)$. Similarly, we define a maximum Y-neighborhood ordering of a graph.

The following theorem relates the graph $G$ which admits a $b$-extremal ordering with the maximum X-neighborhood ordering and maximum Y-neighborhood ordering of $G$.

**Theorem 2.3 ([2]).** Let $G = (V, E)$ be a graph. Then $G$ has a $b$-extremal ordering if and only if $G$ is bipartite, and $G$ has a maximum X-neighborhood ordering and a maximum Y-neighborhood ordering.

### 3. Total Domination in Chordal Bipartite Graphs

Let $G = (X, Y, E)$ be a chordal bipartite graph and $C_X(G)$ (resp. $C_Y(G)$) denote the split graph obtained from $G$ by adding edges between every pair of vertices in $X$ (resp. $Y$). Then $C_X(G)$ is a strongly chordal graph which follows from the following theorem appearing in [1].

**Theorem 3.1 ([1]).** Let $G = (X, Y, E)$ be a bipartite graph. Then $G$ is a chordal bipartite graph if and only if $C_X(G)$ is a strongly chordal graph.

Let $G = (X, Y, E)$ be a bipartite graph. Let $\overline{X}_D$ (resp. $\overline{Y}_D$) be a minimum cardinality subset of $X$ (resp. $Y$) which dominates all the vertices of $Y$ (resp. $X$). Then the following lemma holds.

**Lemma 3.2 ([4]).** $\overline{X}_D \cup \overline{Y}_D$ is a minimum total dominating set of a bipartite graph $G = (X, Y, E)$.

A polynomial time algorithm had been proposed in [4] to compute a minimum total dominating set of a chordal bipartite graph $G = (X, Y, E)$. The idea behind the proposed algorithm is to compute the sets $\overline{X}_D$ and $\overline{Y}_D$ polynomially. The computations of $\overline{X}_D$ and $\overline{Y}_D$ are equivalent to find minimum dominating sets of the graphs $C_X(G)$ and $C_Y(G)$, respectively. By Theorem 3.1, $C_X(G)$ and $C_Y(G)$ are strongly chordal graphs. To compute the sets $\overline{X}_D$ and $\overline{Y}_D$, we need to form the graphs $C_X(G)$ and $C_Y(G)$, respectively. This takes at most $O(n^2)$ time. Again we need to find strong elimination orderings of the graphs $C_X(G)$ and $C_Y(G)$ to apply the algorithm for computing minimum dominating set proposed in [5]. The computation of the strong elimination ordering can be done in $O(\min\{m \log n, n^2\})$ time [16]. So the complexity of computing a minimum total domination set in a chordal bipartite graph is $O(n^2)$ time. Here, we show an easy computations of the sets $\overline{X}_D$ and $\overline{Y}_D$ for chordal bipartite graphs. We need not to form the graphs $C_X(G)$ and $C_Y(G)$.
Proof. Let \( \bar{X}_D \) be the set computed by the \textsc{Algo-Min-Total-Dom}(G, \sigma) after the vertex \( x_i \) is considered. We prove, by induction on \( i \); \( 0 \leq i \leq n_x \) that the set \( Y_D \) is contained in some minimum cardinality subset \( Y_D \) of \( Y \) such that \( Y_D \) dominates all the vertices of \( X \). The base case is trivial as \( Y_D = \emptyset \) is contained in any \( Y_D \). Assume that \( Y_{D_{i-1}} \) is contained in some minimum cardinality subset \( Y'_D \) of \( Y \) such that \( Y'_D \) dominates all the vertices of \( X \). Let \( x_i \) be the considered vertex. If \( N_G(x_i) \cap Y_{D_{i-1}} \neq \emptyset \), then \( Y_{D_i} = Y_{D_{i-1}} \) and we are through. So assume that \( N_G(x_i) \cap Y_{D_{i-1}} = \emptyset \). Let \( y_k \) be the minimum indexed vertex that dominates \( x_i \). Clearly \( y_k \notin Y_{D_{i-1}} \). Again \( k < i^* \). As \( \sigma_X = (x_1, x_2, \ldots, x_{n_x}) \) and \( \sigma_Y = (y_1, y_2, \ldots, y_{n_y}) \) is a strong \( T \)-elimination ordering of \( G \), we have \( N_{G'}(y_k) \subseteq N_G(y_i) \), where \( G' = G[(x_1, x_2, \ldots, x_{n_x}) \cup \{y_1, y_2, \ldots, y_{n_y}\}] \). Now \( Y''_D = (Y'_D \setminus \{y_k\}) \cup \{y_i\} \) is a required minimum cardinality subset of \( Y \) that dominates all the vertices of \( X \). So the

\begin{algorithm}[H]
\caption{\textsc{Algo-Min-Total-Dom}(G, \sigma)}
\begin{algorithmic}[1]
\Input A chordal bipartite graph \( G = (X, Y, E) \) with a weak elimination ordering \( \sigma \);
\Output A minimum total dominating set \( TD \) of \( G \);
\State Compute \( \sigma_X = (x_1, x_2, \ldots, x_{n_x}) \) and \( \sigma_Y = (y_1, y_2, \ldots, y_{n_y}) \), a strong \( T \)-elimination ordering of \( G \);
\State Initialize \( \bar{X}_D = \emptyset \), \( \bar{Y}_D = \emptyset \);
\For {\( i = 1 \) to \( n_x \)}
\If {\( (N_G(x_i) \cap \bar{Y}_D = \emptyset) \)}
\State Let \( i^* = \max\{k|y_k x_i \in E(G)\} \);
\State \( \bar{Y}_D = \bar{Y}_D \cup \{y_{i^*}\} \);
\EndIf
\EndFor;
\For {\( j = 1 \) to \( n_y \)}
\If {\( (N_G(y_j) \cap \bar{X}_D = \emptyset) \)}
\State Let \( j^* = \max\{k|x_k y_j \in E(G)\} \);
\State \( \bar{X}_D = \bar{X}_D \cup \{x_{j^*}\} \);
\EndIf
\EndFor
\State The vertex \( y_k \) that dominates all the vertices of \( Y \) is contained in some minimum cardinality subset \( \bar{Y}_D \) of \( \bar{Y} \).
\end{algorithmic}
\end{algorithm}

To prove the correctness of the algorithm, it is sufficient to prove that the \textsc{Algo-Min-Total-Dom}(G, \sigma) correctly computes the sets \( \bar{X}_D \) and \( \bar{Y}_D \).

Lemma 3.3. \textsc{Algo-Min-Total-Dom}(G, \sigma) correctly computes the sets \( \bar{X}_D \) and \( \bar{Y}_D \).

Proof. Let \( Y_{D_i} \) be the set computed by the \textsc{Algo-Min-Total-Dom}(G, \sigma) after the vertex \( x_i \) is considered. We prove, by induction on \( i \); \( 0 \leq i \leq n_x \) that the set \( Y_{D_i} \) is contained in some minimum cardinality subset \( Y_D \) of \( Y \) such that \( Y_D \) dominates all the vertices of \( X \). The base case is trivial as \( Y_D = \emptyset \) is contained in any \( Y_D \). Assume that \( Y_{D_{i-1}} \) is contained in some minimum cardinality subset \( Y'_D \) of \( Y \) such that \( Y'_D \) dominates all the vertices of \( X \). Let \( x_i \) be the considered vertex. If \( N_G(x_i) \cap Y_{D_{i-1}} \neq \emptyset \), then \( Y_{D_i} = Y_{D_{i-1}} \) and we are through. So assume that \( N_G(x_i) \cap Y_{D_{i-1}} = \emptyset \). Let \( y_k \) be the minimum indexed vertex that dominates \( x_i \). Clearly \( y_k \notin Y_{D_{i-1}} \). Again \( k < i^* \). As \( \sigma_X = (x_1, x_2, \ldots, x_{n_x}) \) and \( \sigma_Y = (y_1, y_2, \ldots, y_{n_y}) \) is a strong \( T \)-elimination ordering of \( G \), we have \( N_{G'}(y_k) \subseteq N_G(y_i) \), where \( G' = G[(x_1, x_2, \ldots, x_{n_x}) \cup \{y_1, y_2, \ldots, y_{n_y}\}] \). Now \( Y''_D = (Y'_D \setminus \{y_k\}) \cup \{y_i\} \) is a required minimum cardinality subset of \( Y \) that dominates all the vertices of \( X \). So the
induction hypothesis is true which implies that $\overline{X_D}$ is a minimum cardinality subset of $Y$ that dominates all the vertices of $X$.

Similarly, by induction, we can prove that $\overline{X_D}$ is a minimum cardinality subset of $X$ that dominates all the vertices of $Y$.

By Lemma 3.3, $\overline{X_D} \cup \overline{Y_D}$ is a minimum total dominating set of $G$. Again by Lemma 2.2, given a weak elimination ordering of $G$, a strong $T$-elimination ordering of $G$ can be computed in $O(n)$ time. Note that the algorithm scans the vertices and edges of the graph constant number of times. So we have the following theorem.

**Theorem 3.4.** Given a weak elimination ordering of a chordal bipartite graph $G = (X, Y, E)$, a minimum total dominating set of $G$ can be computed in $O(n + m)$ time.

4. Signed and Minus Total Domination in Chordal Bipartite Graphs

In this section, we present our approach to find a minimum signed (resp. minus) total dominating function in chordal bipartite graphs. For this, we introduce $R$-total domination in graphs which is also introduced in [15].

Let $\ell, d, I_1$ be fixed integers and $\ell, d > 0$. Let $P$ be the weight set $\{I_1, I_1 + d, \ldots, I_1 + (\ell - 1) \cdot d\}$. Suppose that $G = (X, Y, E)$ be a bipartite graph with a labeling function $R$ which assigns an integer $R(v)$ to each $v \in V(G)$. A function $f : V \rightarrow P$ is called a $R$-total dominating function if $f(N_G(v)) \geq R(v)$ for each $v \in V$. Note that a $k$-tuple total dominating set can be viewed as a $R$-total dominating set with $R = k$ and $P = \{0, 1\}$. Similarly a signed (resp. minus) total dominating function can be viewed as a $R$-total dominating function with $R = 1$ and $P = \{-1, 1\}$ (resp. $R = 1$ and $P = \{-1, 0, 1\}$).

Let $R_X$ (resp. $R_Y$) be a labeling function of $G$ which assigns an integer $R_X(v)$ (resp. $R_Y(v)$) to each vertex in $G$ such that $R_X(v) = I_1 \cdot d_G(v)$ (resp. $R_Y(v) = I_1 \cdot d_G(v)$) for each $v \in X$ (resp. $v \in Y$), and $R_X(v) = R(v)$ (resp. $R_Y(v) = R(v)$) for every $v \in Y$ (resp. $v \in X$).

An $R_X$-total dominating function $f$ of a bipartite graph $G = (X, Y, E)$ is called an $R_X^\ast$-total dominating function of $G$ if $f(v) = I_1 + (\ell - 1) \cdot d$ for every $v \in Y$. An $R_Y$-total dominating function $f$ of a bipartite graph $G = (X, Y, E)$ is called an $R_Y^\ast$-total dominating function of $G$ if $f(v) = I_1 + (\ell - 1) \cdot d$ for every $v \in X$.

The following lemma shows that a minimum $R$-total dominating function of a bipartite graph $G$ can be obtained from a minimum $R_X^\ast$-total dominating function and a minimum $R_Y^\ast$-total dominating function of $G$.

**Lemma 4.1 ([15]).** Suppose that $G = (X, Y, E)$ is a bipartite graph with a labeling function $R$ as mentioned above. Let $f_X$ (resp. $f_Y$) be a minimum $R_X^\ast$-total (resp. $R_Y^\ast$-total) dominating function of $G$. Let $f$ be a function of $G$ defined by $f(v) = f_X(v)$ for every $v \in X$ and $f(v) = f_Y(v)$ for every $v \in Y$. Then $f$ is a minimum $R$-total dominating function of $G$. 

1250045-8
Complexity of Certain Functional Variants of Total Domination in Chordal Bipartite Graphs

Based on the Lemma 4.1, we now present the algorithm, namely \textsc{MIN-RTDSBip}(G, R, I_1, \ell, d) that computes the functions \(f_X\) and \(f_Y\) in \(O(n + m)\) time. \textsc{MIN-RTDSBip}(G, R, I_1, \ell, d) takes a chordal bipartite \(G = (X, Y, E)\) with a given weak elimination ordering \(\sigma = (v_1, v_2, \ldots, v_n)\) of \(X \cup Y\), the function \(R, I_1, \ell\) and \(d\) as the inputs. The algorithm is presented below.

\textbf{Algorithm 2:} \textsc{MIN-RTDSBip}(G, R, I_1, \ell, d)

\begin{algorithm}
\textbf{Input:} A chordal bipartite graph \(G = (X, Y, E)\) with a weak elimination ordering \(\sigma = (v_1, v_2, \ldots, v_n)\) of \(X \cup Y\);
\textbf{Output:} A minimum \(R_X^\delta\)-total dominating function of \(G\);
1. Compute a strong \(T\)-elimination ordering \(\sigma_X = (x_1, x_2, \ldots, x_m)\) and \(\sigma_Y = (y_1, y_2, \ldots, y_n)\) as shown in Lemma 2.1;
2. for \(i = 1\) to \(n_x\) do
3. \hspace{1em} \(R_X(x_i) = I_1 \cdot d_G(x_i)\);
4. end
5. for \(i = 1\) to \(n_y\) do
6. \hspace{1em} \(R_Y(y_i) = R(y_i)\);
7. end
8. \hspace{1em} Initialize \(f(v) = I_1 + (\ell - 1) \cdot d\) for every vertex \(v \in X \cup Y\);
9. if \((R_X(v) > f(N_G(v))\) for any \(v \in X \cup Y)\) then
10. \hspace{1em} Output “Stop and No Feasible Solution”;
11. end
12. for \(i = 1\) to \(n_x\) do
13. \hspace{1em} \(M = \min\{f(N_G(v)) - R_X(v)\vert v \in N_G(x_i)\}\);
14. \hspace{1em} \(f(x_i) = \max\{I_1, I_1 + (\lceil \ell - \frac{M}{d} \rceil - 1) \cdot d\}\);
15. end
16. Output the function \(f\);
\end{algorithm}

We now prove the correctness of the algorithm \textsc{MIN-RTDSBip}(G, R, I_1, \ell, d) in Lemmas 4.2–4.4.

\textbf{Lemma 4.2.} Suppose the function \(f\) is initialized as \(f(v) = I_1 + (\ell - 1) \cdot d\) for every vertex \(v \in X \cup Y\). If \(f\) is not an \(R_X^\delta\)-total dominating function of \(G\), then \(G\) has no \(R\)-total dominating function.

\textbf{Proof.} Note that \(f\) has assigned the maximum value of \(\mathcal{P}\) to every vertex of \(G\). So \(f\) has maximum weight among all \(R_X^\delta\)-total (resp. \(R\)-total) dominating functions of \(G\) if \(f\) is an \(R_X^\delta\)-total (resp. \(R\)-total) dominating function of \(G\). Since minimum value of \(\mathcal{P}\) is \(I_1\), \(f(N_G(v)) > R_X(v)\) for every \(v \in X\). If there is a vertex \(u \in X \cup Y\) with \(R_X(u) > f(N_G(u))\), then \(u \in Y\). So \(R(u) = R_X(u) > f(N_G(u))\). This implies that \(f\) is neither an \(R_X^\delta\)-total dominating function nor an \(R\)-total dominating function of \(G\). Therefore, \(G\) has no \(R\)-total dominating function. \(\square\)
The smallest index such that $x_{i}$. We claim that $f$ is true. We assume that at the end of (iteration) $i$ of the “for” loop of Step 12, the updated function $f$ obtained in Step 14 is still an $R^*_X$-total dominating function of $G$.

Let $i = 0$. Note that at the beginning of the first iteration of the “for” loop of Step 12, $f$ is an $R^*_X$-total dominating function by Lemma 4.2. So the base case is true. We assume that at the end of $(i-1)$th iteration of “for” loop of Step 12, the function $f$ is an $R^*_X$-total dominating function of $G$. Suppose that $x_i$ is the vertex such that $f$ is updated for $x_i$ at the end of $i$th iteration of “for” loop of Step 12. Let $x = \max\{I_1, I_1 + (\lceil \ell - \frac{M}{d} \rceil - 1) \cdot d\}$. Then $x \geq I_1 + (\lceil \ell - \frac{M}{d} \rceil - 1) \cdot d$. We have

$$x \geq I_1 + (\ell - 1 - \frac{M}{d}) \cdot d \implies M \geq (I_1 + (\ell - 1) \cdot d) - x.$$  

Since $M = \min\{f(N_G(v)) - R_X(v) \mid v \in N_G(x_i)\}$, $f(N_G(v)) - R_X(v) \geq (I_1 + (\ell - 1) \cdot d) - x$ for every vertex $v \in N_G(x_i)$. Now we have $f(N_G(v)) - (I_1 + (\ell - 1) \cdot d) + x \geq R_X(v)$. Note that $f(x_i) = I_1 + (\ell - 1) \cdot d$ at the beginning of the $i$th iteration of the “for” loop of Step 12. Therefore, the updated function $f$ obtained by replacing the value of $f(v)$ with $x$ is still an $R^*_X$-total dominating function of $G$. Therefore, the output function $f$ of MIN-RTDSBip($G, R, I_1, \ell, d$) is an $R^*_X$-total dominating function of $G$.

**Lemma 4.4.** Let $G = (X, Y, E)$ be a chordal bipartite graph and $\ell, d, I_1$ be the integers such that $\ell, d > 0$. Let $P$ be the set $\{I_1, I_1 + d, \ldots, I_1 + (\ell - 1) \cdot d\}$. The function $f : X \cup Y \to P$ returned by MIN-RTDSBip($G, R, I_1, \ell, d$) is a minimum $R^*_X$-total dominating function of $G$.

**Proof.** By Lemma 4.3, it is clear that $f$ is an $R^*_X$-total dominating function of $G$. Now we show that $f$ is a minimum $R^*_X$-total dominating function of $G$. Among all minimum $R^*_X$-total dominating functions of $G$, let $h$ be a minimum $R^*_X$-total dominating function of $G$ such that $\{|v \mid v \in X \cup Y, f(v) = h(v)\}$ is maximum. We claim that $f(v) = h(v)$ for every $v \in X \cup Y$. Assume on the contrary that $W = \{w \in X \cup Y \mid f(w) \neq h(w)\}$ is nonempty. Note that $W \subseteq X$. Let $t$ be the smallest index such that $x_t \in W \cap X$. We need to consider the following cases:

**Case 1:** $h(x_t) < f(x_t)$.

It is clear that $f(x_t) = \max\{I_1, I_1 + (\lceil \ell - \frac{M}{d} \rceil - 1) \cdot d\}$ at the end of $t$th iteration of the “for” loop of Step 12. Here $M = \min\{f(N_G(v)) - R_X(v) \mid v \in N_G(x_t)\}$. Depending on $W$, we have the following possibilities:

- **Case 1a:** $W \cap X = \emptyset$.

- **Case 1b:** $W \cap X \neq \emptyset$.

on whether $f(x_t) = I_1$ or $f(x_t) = I_1 + ([\ell - \frac{M}{d}] - 1) \cdot d$, we have the following two subcases.

**Subcase 1.1: $f(x_t) = I_1$.**

Now we have $h(x_t) < f(x_t) = I_1$. This contradicts the fact that $h(x_t) \in \mathcal{P}$ since $I_1$ is the smallest integer in $\mathcal{P}$.

**Subcase 1.2: $f(x_t) = I_1 + ([\ell - \frac{M}{d}] - 1) \cdot d$.**

It is clear that $h(x_t) \leq f(x_t) - d = I_1 + ([\ell - \frac{M}{d}] - 2) \cdot d$. Let $y_c \in N_G(x_t)$ such that $M = f(N_G(y_c)) - R_X(y_c)$. Note that $f(x_t) = I_1 + (\ell - 1) \cdot d$ at the beginning of $t$th iteration of the “for” loop of Step 12. So, $f(N_G(y_c) \setminus \{x_t\}) = M + R_X(y_c) - (I_1 + (\ell - 1) \cdot d)$ before the execution of Step 14 at the $t$th iteration. Notice that the value $f(x_t)$ is only updated at the $t$th iteration of the “for” loop of Step 12. Therefore, the value of $f(N_G(y_c) \setminus \{x_t\})$ still remains $M + R_X(y_c) - (I_1 + (\ell - 1) \cdot d)$ at the end of $t$th iteration of the “for” loop of Step 12.

Again we have $h(x_r) = f(x_r)$ for every index $r < t$. At the end of $t$th iteration of the “for” loop of Step 12, $h(x_r) \leq f(x_r) = I_1 + (\ell - 1) \cdot d$ for every index $r > t$. Then,

$$h(N_G(y_c)) \leq f(N_G(y_c) \setminus \{x_t\}) + h(x_t)$$

$$\leq f(N_G(y_c) \setminus \{x_t\}) + I_1 + \left(\left[\ell - \frac{M}{d}\right] - 2\right) \cdot d$$

$$= M + R_X(y_c) - (I_1 + (\ell - 1) \cdot d) + I_1 + \left(\left[\ell - \frac{M}{d}\right] - 2\right) \cdot d$$

$$\leq M + R_X(y_c) - (I_1 + (\ell - 1) \cdot d) + I_1 + \left(\left[\ell - \frac{M}{d}\right] - 1\right) \cdot d$$

$$= M + R_X(y_c) - (I_1 + (\ell - 1) \cdot d) + I_1 + (\ell - 1) \cdot d - M$$

$$= R_X(y_c).$$

Therefore, $h(N_G(y_c)) < R_X(y_c)$. This contradicts the fact that $h$ is an $R_X^*$-total dominating function of $G$.

**Case 2: $f(x_t) < h(x_t)$.**

Let $\mathcal{P} = \{p_1, p_2, \ldots, p_t\}$, where $p_i = I_1 + (i - 1) \cdot d$ for $1 \leq i \leq \ell$. Let $f(x_t) = p_j$ and $h(x_t) = p_k$ for $1 \leq j < k \leq \ell$. Let $Y^* = \{y | y \in N_G(x_t) \text{ and } h(N_G(y)) - p_k + p_j < R_X(y)\} \subseteq Y$. Note that $Y^* \neq \emptyset$; otherwise $h(N_G(y)) - p_k + p_j$ for every $y \in N_G(x_t)$ and there is an $R_X^*$-total dominating function $g$ with $g(Y) < h(Y)$ by setting $g(x_t) = h(x_t) - p_k + p_j$ and $g(u) = h(u)$ for every vertex $u \in (X \cup Y) \setminus \{x_t\}$, which contradicts the fact that $h$ is a minimum $R_X^*$-total dominating function of $G$.

Notice that $h(x_r) = f(x_r)$ for every index $r < t$. Since $h(N_G(y)) - p_k + p_j < R_X(y)$ and $f(N_G(y)) \geq R_X(y)$ for every vertex $y \in Y^*$, we have $N_G(y) \cap \{x_s | x_s \in W, t < s, \text{ and } h(x_s) < f(x_s)\} \neq \emptyset$. Let $X^*(y) = N_G(y) \cap \{x_s | x_s \in W, t < s, \text{ and } h(x_s) < f(x_s)\}$ for every vertex $y \in Y^*$. Clearly $X^*(y) \subseteq X$ for every $y \in Y^*$.
Let \( q \) be the smallest index such that \( y_q \in Y^* \) and \( b \) be the smallest index such that \( x_b \in X^*(y_q) \). Note that \( Y^* \subseteq N_G(x_1) \subseteq Y \). Since \( h(x_t), f(x_t), h(x_b), f(x_b) \in \mathcal{P} \), there exist two positive integers \( c_1 \) and \( c_2 \) such that \( h(x_t) = f(x_t) + c_1 \cdot d \) and \( f(x_b) = h(x_b) + c_2 \cdot d \). Define a function \( h' \) as follows:

- If \( c_1 \leq c_2 \), \( h'(x_t) = h(x_t) - c_1 \cdot d = f(x_t), h'(x_b) = h(x_b) + c_1 \cdot d \) and \( h'(u) = h(u) \) for every vertex \( u \in (X \cup Y) \setminus \{x_t, x_b\} \).
- If \( c_1 > c_2 \), \( h'(x_t) = h(x_t) - c_2 \cdot d = f(x_b) \) and \( h'(u) = h(u) \) for every vertex \( u \in (X \cup Y) \setminus \{x_t, x_b\} \).

Clearly, \( h(X \cup Y) = h'(X \cup Y) \) and \(|\{u | u \in X \cup Y, f(u) = h'(u)\}| \geq |\{u | u \in X \cup Y, f(u) = h(u)\}| + 1\). We prove that \( h'(N_G(u)) \geq R_X(u) \) for every \( u \in X \cup Y \) by showing that \( Y^* \subseteq N_G(x_b) \).

**Claim:** \( Y^* \subseteq N_G(x_b) \).

**Proof of the Claim.** Since \( y_q \in Y^* \) and \( x_b \in X^*(y_q) \), \( y_q \in Y \) and \( x_1, x_b \in X \). Again \( q \neq t \). Notice that \( t < b \). Since \( \sigma_X = (x_1, x_2, \ldots, x_{n_x}) \) and \( \sigma_Y = (y_1, y_2, \ldots, y_{n_y}) \) is a strong \( T \)-elimination ordering of \( G \) obtained from the given weak elimination ordering \( \sigma \) of \( G \), by Lemma 2.2, \( N_G'(x_1) \subseteq N_G'(x_b) \), where \( G' = G[\{y_1, y_{q+1}, \ldots, y_{n_y}\} \cup \{x_1, x_2, \ldots, x_{n_x}\}] \). Since \( Y^* \subseteq N_G'(x_t) \), we have \( Y^* \subseteq N_G(x_t) \subseteq N_G(x_b) \). \( \blacksquare \)

Therefore, \( h' \) is a minimum \( R_X^* \)-total dominating function of \( G \) such that \(|\{u | u \in X \cup Y, f(u) = h'(u)\}| > |\{u | u \in X \cup Y, f(u) = h(u)\}| \) which contradicts the choice of the minimum \( R_X^* \)-total dominating function \( h \).

Now we can conclude from above discussions that \( W \) does not exist. Therefore, \( f \) is a minimum \( R_X^* \)-total dominating function of \( G \). \( \blacksquare \)

**Theorem 4.5.** Let \( G = (X, Y, E) \) be a chordal bipartite graph with \(|X \cup Y| = n\) and \(|E| = m\). A minimum \( R \)-total dominating function \( f \) of \( G \) can be computed in \( O(n + m) \)-time if a weak elimination ordering of \( G \) is provided.

**Proof.** Given a weak elimination ordering of \( G \), a strong \( T \)-elimination ordering of \( G \) as shown in Lemma 2.2 can be computed in \( O(n) \) time. By Lemma 4.4, it clear that \( \text{MIN-RTDSBip}(G, R, I, t, \ell, d) \) computes a minimum \( R_X^* \)-total (resp. \( R_Y^* \)-total) dominating function of \( G \). The rest is to show that \( \text{MIN-RTDSBip}(G, R, I, t, \ell, d) \) takes at most \( O(n + m) \) time.

To maintain the values of \( f(N_G(v_j)) \) and \( f(N_G(v_i)) - R_X(v_i) \), we use two arrays \( D[1, \ldots, n] \) and \( A[1, \ldots, n] \), respectively. For the initialization at Step 8, we initialize \( D[i] = (I_1 + (\ell - 1) \cdot d) \cdot d_G(v_i) \) and \( A[i] = D[i] - R_X(v_i) \) for each \( i \). This can be done in at most \( O(d_G(v_i) + 1) \) time for each \( i \). So the initialization of the arrays \( D \) and \( A \) can be done in \( O\left(\sum_{i=1}^{n}(d_G(v_i) + 1)\right) = O(n + m) \) time. When \( f(v_i) \) is updated to a number \( x \in \mathcal{P} \), the values of \( D[j] \) and \( A[j] \) are respectively decreased by \( (I_1 + (\ell - 1) \cdot d) - x \) for each vertex \( v_j \in N_G(v_i) \). So this update can be done in
O(d_G(v_i) + 1) time. At ith iteration of the “for” loop of Step 12, 1 ≤ i ≤ n, the values of M can be computed by verifying the values of A[j] for each vertex v_j ∈ N_G(v_i). So throughout the algorithm Min-RTDSBip(G, R, I_1, ℓ, d), all the required values of M can be computed in at most O(\sum_{i=1}^{n}(d_G(v_i) + 1)) = O(n + m) time. Therefore, Min-RTDSBip(G, R, I_1, ℓ, d) takes at most O(n + m) time and hence a minimum R-total dominating function f of G can be computed in O(n + m) time.

5. Complexity Results

5.1. Complexity of Min-Total-Dom-Set in perfect elimination bipartite graphs

Min-Total-Dom-Set is NP-complete for bipartite graphs [17]. In this subsection, we strengthen this result by showing that Min-Total-Dom-Set is NP-complete for perfect edge elimination bipartite graphs. We do this by providing a polynomial time reduction of Min-Total-Dom-Set problem in bipartite graphs to Min-Total-Dom-Set problem in perfect elimination bipartite graphs.

Theorem 5.1. Min-Total-Dom-Set is NP-complete for perfect edge elimination bipartite graphs.

Proof. Clearly Min-Total-Dom-Set problem for perfect elimination bipartite graphs is in NP.

Let G = (X, Y, E) be a bipartite graph. We construct a bipartite graph G’ by first attaching a path y_i - a_i - b_i - d_i - e_i to each v_i ∈ X ∪ Y and then attaching the path b_i - c_i to b_i for all i, 1 ≤ i ≤ n. The construction of G’ from G is illustrated in Fig. 1.

Note that G’ can be constructed in polynomial time given G. Clearly G’ is a bipartite graph. Now (e_1d_1, e_2d_2, ..., e_nd_n, c_1b_1, c_2b_2, ..., c_nb_n, a_1v_1, a_2v_2, ..., a_nv_n) is a perfect edge elimination ordering of G’. Hence G’ is a perfect elimination bipartite graph.

Fact 1: G’ has a minimum total dominating set, say D’ such that b_i, d_i ∈ D’ for all i, 1 ≤ i ≤ n.

Fig. 1. Illustration of the construction of G’.
Claim: $\gamma_t(G') = \gamma_t(G) + 2n$.

Proof the Claim. Let $D$ be a minimum total dominating set of $G$. Then $D \cup \{b_i, d_i | 1 \leq i \leq n\}$ is a total dominating set of $G'$. So $\gamma_t(G') \leq \gamma_t(G) + 2n$.

Among all minimum total dominating sets of $G'$, consider a minimum total dominating set, say $D_1$, of $G'$ such that $b_i, d_i \in D_1$ for all $1 \leq i \leq n$. Let $v_i \in V(G)$ be some vertex. If $(N_{G'}(v_i) \cap D_1 \setminus \{a_i\}) \neq \emptyset$, then $D_1 \setminus \{a_i\}$ is a smaller total dominating set of $G$ which is a contradiction. So $N_{G'}(v_i) \cap D_1 = \{a_i\}$. Then let $D'_1 = (D_1 \setminus \{a_i\}) \cup \{u\}$, where $u \in N_{G'}(v_i)$. Note that such a vertex exists and is adjacent to a vertex $u'$ in $D_1$, in particular $u' \in N_{G'}(u) \cap D_1$. So without loss of generality, we take $a_i \notin D_1$ for all $1 \leq i \leq n$. Now $D_1 \setminus \{b_i, d_i | 1 \leq i \leq n\}$ is a total dominating set of $G$ and hence $\gamma_t(G) \leq \gamma_t(G') - 2n$ which implies the equality.

Hence Min-Total-Dom-Set problem is NP-complete for perfect elimination bipartite graphs.

5.2. Complexity of Min-Signed-Total-Dom-Function in perfect elimination bipartite graphs and graphs with a b-extremal ordering

In this section, we show that Min-Signed-Total-Dom-Function problem is NP-complete for perfect elimination bipartite graphs and for graphs with a b-extremal ordering.

Min-Signed-Total-Dom-Function is shown to be NP-complete for bipartite graphs [12]. The bipartite graph constructed in [12] is also a perfect edge elimination bipartite graphs. So we have the following theorem.

Theorem 5.2. Min-Signed-Total-Dom-Function is NP-complete for perfect edge elimination bipartite graphs.

Let $G = (V, E)$ be a graph. A function $f : V \rightarrow \{-1, 1\}$ is a signed total zero dominating function if $f(N_G(v)) \geq 0$ for every $v \in V$. Given a graph $G = (V, E)$ and an integer $l > 0$, in Min-Signed-Total-Zero-Dom-Function problem, it is required to decide whether $G$ has a signed total dominating function of weight at most $l$. The following theorem is taken from [15] which will be used later.

Theorem 5.3 ([15]). Min-Signed-Total-Zero-Dom-Function is NP-complete for chordal graphs.

Note that the construction given in proof of Theorem 5.3 is also valid for bipartite graphs i.e., if we start from the Min-Total-Dom-Set problem in bipartite graphs
Complexity of Certain Functional Variants of Total Domination in Chordal Bipartite Graphs

(known to be NP-complete [17]), then we can show that \textsc{Min-Signed-Total-Zero-Dom-Function} is NP-complete for bipartite graphs. We record this in the following corollary for further use.

\textbf{Corollary 5.4.} \textsc{Min-Total-Signed-Zero-Dom-Function} is \textsc{NP}-complete for bipartite graphs.

\textbf{Theorem 5.5.} \textsc{Min-Signed-Total-Dom-Function} is \textsc{NP}-complete for graphs with a \textit{b}-extremal ordering.

\textbf{Proof.} Note that \textsc{Min-Signed-Total-Dom-Function} problem for graphs with a \textit{b}-extremal ordering is in \textsc{NP}. By Corollary 5.4, \textsc{Min-Signed-Total-Zero-Dom-Function} is \textsc{NP}-complete for bipartite graphs. In the following, we show the \textsc{NP}-completeness of \textsc{Min-Signed-Total-Dom-Function} problem for graphs with a \textit{b}-extremal ordering by providing a polynomial time reduction from \textsc{Min-Signed-Total-Zero-Dom-Function} for bipartite graphs to it.

Given a bipartite graph \(G = (X, Y, E)\) and \(x, y \notin X \cup Y\), we construct a bipartite graph \(G' = (X', Y', E')\), where \(X' = X \cup \{x\} \cup \{u_i, z_j | 1 \leq i \leq |X| + 1, 1 \leq j \leq |Y| + 1\}\), \(Y' = Y \cup \{y\} \cup \{v_j, w_i | 1 \leq j \leq |Y| + 1, 1 \leq i \leq |X| + 1\}\) and \(E' = E \cup \{xy\} \cup \{xv_i, z_j, w_i, v_j | 1 \leq i \leq |X| + 1, u \in X\} \cup \{yu, u_i, w_i, v_j | 1 \leq i \leq |X| + 1\}\).

Note that \(G'\) has a maximum \(X'\)-neighborhood ordering as well as a maximum \(Y'\)-neighborhood ordering and hence by Theorem 2.3, \(G\) is a graph that admits a \textit{b}-extremal ordering. Since \(|V(G')| = 3|V(G)| + 6\), the construction of \(G'\) can be done in polynomial time.

For a graph \(G\), let \(\gamma_{0,t,s}^0(G)\) be the minimum weight of a signed total zero dominating function of \(G\).

\textbf{Claim:} \(\gamma_{t,s}^0(G') = \gamma_{t,s}^0(G) + (2n + 6)\).

\textbf{Proof of the Claim.} Suppose that \(g\) is a minimum signed total zero dominating function of \(G\). Let \(h: V(G') \to \{-1, 1\}\) be a function of \(G'\) defined by \(h(x) = h(y) = h(u_i) = h(w_i) = h(v_j) = 1\) for each \(1 \leq i \leq |X| + 1; 1 \leq j \leq |Y| + 1\) and \(h(v) = g(v)\) if \(v \in V(G)\). It is clear that \(h\) is a signed total dominating function of \(G'\) and hence we have \(\gamma_{t,s}^0(G') \leq \gamma_{t,s}^0(G) + (2n + 6)\).

Assume that \(h\) is a minimum signed total dominating function of \(G'\). Then it is clear that \(h(x) = h(v_j) = h(z_j) = 1\) for each \(1 \leq j \leq |Y| + 1\). Similarly \(h(y) = h(u_i) = h(w_i) = 1\) for each \(1 \leq i \leq |X| + 1\). Let \(g: V(G) \to \{-1, 1\}\) such that \(g(v) = h(v)\) for every \(v \in V(G)\). By construction of the graph \(G'\), every vertex \(a \in X\) is adjacent to \(y\). Therefore, \(g(N_G(a)) = h(N_{G'}(a)) - 1 \geq 0\). Similarly, we get \(g(N_G(b)) = h(N_{G'}(b)) - 1 \geq 0\) for every \(b \in Y\). So, the function \(g\) is a signed total zero dominating function of \(G\). Hence \(\gamma_{t,s}^0(G) \leq \gamma_{t,s}^0(G') - (2n + 6)\) and this completes the proof of the claim.

Therefore, \textsc{Min-Signed-Total-Dom-Function} is \textsc{NP}-complete for graphs with a \textit{b}-extremal ordering.
5.3. **Complexity of Min-\(k\)-Tuple-Total-Dom-Set for perfect elimination bipartite graphs and graphs with a \(b\)-extremal orderings**

**Theorem 5.6.** For any fixed integer \(k > 1\), Min-\(k\)-Tuple-Total-Dom-Set is \(\text{NP-complete for graphs with a } b\text{-extremal ordering.}\)

**Proof.** Note that the Min-\(k\)-Tuple-Total-Dom-Set problem for graphs with a \(b\)-extremal ordering is in \(\text{NP}\). We next show the \(\text{NP-completeness of Min-}\(k\)-Tuple-Total-Dom-Set problem for graphs with a \(b\)-extremal ordering by giving a polynomial time reduction from Min-(\(k-1\))-Tuple-Total-Dom-Set problem for bipartite graphs to it. Note that for \(k = 2\), Min-1-Tuple-Total-Dom-Set i.e., Min-Total-Dom-Set is \(\text{NP-complete for bipartite graphs}\) [17].

Given a bipartite graph \(G = (X, Y, E)\), an instance of Min-(\(k-1\))-Tuple-Total-Dom-Set problem for bipartite graphs, we construct a bipartite graph \(G' = (X', Y', E')\) as follows:

- Introduce two new adjacent vertices \(x\) and \(y\) such that \(xv \in E'\) for all \(v \in Y\) and \(yu \in E'\) for all \(u \in X\).
- Introduce two complete bipartite graphs \(G_1 = K_{k-1,k} = (X_1, Y_1, E_1)\) and \(G_2 = K_{k-1,k} = (X_2, Y_2, E_2)\) such that \(xv \in E'\) for all \(v \in Y_1\) and \(yu \in E'\) for all \(u \in Y_2\).

Note that \(X' = X \cup X_1 \cup Y_2 \cup \{x\}\) and \(Y' = Y \cup X_2 \cup Y_1 \cup \{y\}\). It is clear that \(G'\) is a graph with a \(b\)-extremal ordering as \(G'\) has a maximum \(X'\)-neighborhood ordering as well as a maximum \(Y'\)-neighborhood ordering. Since \(|V(G')| = |V(G)| + 4k\), the construction of \(G'\) can be done in polynomial time.

**Claim:** \(\gamma_{\times k,t}(G') = \gamma_{\times (k-1),t}(G) + 4k\).

**Proof of the Claim.** If \(D\) is a minimum \((k-1)\)-tuple total dominating set of \(G\), then \(D^* = D \cup (X_1 \cup X_2 \cup Y_1 \cup Y_2) \cup \{x, y\}\) is a \(k\)-tuple total dominating set of \(G'\). So \(\gamma_{\times k,t}(G') \leq |D| + 4k = \gamma_{\times (k-1),t}(G) + 4k\).

Now assume that \(D^*\) is a minimum \(k\)-tuple total dominating set of \(G\). Then it is clear that \((X_1 \cup X_2 \cup Y_1 \cup Y_2) \cup \{x, y\} \subseteq D^*\). Now let \(D^{**} = D^* \setminus (X_1 \cup X_2 \cup Y_1 \cup Y_2 \cup \{x, y\})\). Since each vertex \(a \in X\) is adjacent to \(y\), \(|N_G(a) \cap D^{**}| = |N_{G'}(a) \cap D^*| - 1 \geq k - 1\). Similarly, we have \(|N_G(b) \cap D^{**}| \geq k - 1\) for each vertex \(b \in Y\). This implies that \(D^{**}\) is a \((k-1)\)-tuple total dominating set of \(G\). Hence \(\gamma_{\times (k-1),t}(G) \leq \gamma_{\times k,t}(G') - 4k\) and this completes the proof of the claim.

Therefore, Min-\(k\)-Tuple-Total-Dom-Set is \(\text{NP-complete for graphs with } b\text{-extremal ordering.}\)

Note that if \(G\) is a perfect edge elimination bipartite graph, then the bipartite graph \(G'\) constructed in Theorem 5.6 is also a perfect edge elimination bipartite
Complexity of Certain Functional Variants of Total Domination in Chordal Bipartite Graphs

graph. Since Min-Total-Dom-Set is NP-complete for perfect edge elimination bipartite graphs by Theorem 5.1, we have the following Corollary.

**Corollary 5.7.** For any fixed integer \( k \geq 1 \), Min-\( k \)-Tuple-Total-Dom-Set is NP-complete for perfect edge elimination bipartite graphs.

5.4. **Complexity of Min-Minus-Total-Dom-Function problem in perfect edge elimination bipartite graphs**

**Theorem 5.8.** Min-Minus-Total-Dom-Function is NP-complete for perfect edge elimination bipartite graphs.

**Proof.** It is clear that Min-Minus-Total-Dom-Function problem for perfect edge elimination bipartite graphs is in NP. Next we show the NP-completeness of Min-Minus-Total-Dom-Function problem for perfect edge elimination bipartite graphs by providing a polynomial time reduction from Min-Total-Dom-Set problem for perfect edge elimination bipartite graphs to it.

Let \( G = (X, Y, E) \) be a perfect edge elimination bipartite graph and \((x_1 y_1, x_2 y_2, \ldots, x_k y_k)\) be a perfect edge elimination ordering of \( G \). We construct a bipartite graph by attaching a path \( a_i - b_i - c_i - d_i \) to each vertex \( v_i \in X \cup Y \). Note that \( G' \) is a perfect edge elimination bipartite graph as \((d_1 c_1, \ldots, d_n c_n, b_1 a_1, \ldots, b_n a_n, x_1 y_1, \ldots, x_k y_k)\) is a perfect edge elimination ordering of \( G' \). This construction was used to prove that Min-Minus-Total-Dom-Function problem is NP-complete for bipartite and chordal graphs \([8]\). So we have \( \gamma_{t,-}(G') = \gamma_t(G) + 2n \). Hence Min-Minus-Total-Dom-Function problem is NP-complete for perfect edge elimination bipartite graphs.

6. **Conclusion**

In this paper, we have presented an algorithm that computes a minimum total dominating set in a chordal bipartite graph \( G = (V, E) \) in \( O(n + m) \) time, where \( n = |V| \) and \( m = |E| \) if a weak elimination ordering of \( G \) is provided. We have then given an algorithm that computes a minimum signed (minus) total dominating function of a chordal bipartite graph in \( O(n + m) \) time if a weak elimination ordering of \( G \) is provided. On the negative side, we have presented NP-completeness results of the above problems in perfect elimination bipartite graphs and in graphs with a \( b \)-extremal ordering.

**Acknowledgments**

The author would like to thank the anonymous referee for his helpful comments leading to improvements in the presentation of the paper.
D. Pradhan

References

Complexity of Certain Functional Variants of Total Domination in Chordal Bipartite Graphs

