Complexity of distance paired-domination problem in graphs

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\textbf{ABSTRACT}

Suppose $G = (V, E)$ is a simple graph and $k$ is a fixed positive integer. A subset $D \subseteq V$ is a distance $k$-dominating set of $G$ if for every $u \in V$, there exists a vertex $v \in D$ such that $d_G(u, v) \leq k$, where $d_G(u, v)$ is the distance between $u$ and $v$ in $G$. A set $D \subseteq V$ is a distance $k$-paired-dominating set of $G$ if $D$ is a distance $k$-dominating set and the induced subgraph $G[D]$ contains a perfect matching. Given a graph $G = (V, E)$ and a fixed integer $k > 0$, the Min Distance $k$-Paired-Dom Set problem is to find a minimum cardinality distance $k$-paired-dominating set of $G$. In this paper, we show that the decision version of Min Distance $k$-Paired-Dom Set is NP-complete for undirected path graphs. This strengthens the complexity of decision version of Min Distance $k$-Paired-Dom Set problem in chordal graphs. We show that for a given graph $G$, unless $NP \subseteq DTIME(n^{O(1/\varepsilon)})$, Min Distance $k$-Paired-Dom Set problem cannot be approximated within a factor of $(1 - \varepsilon)$ in $n$ for any $\varepsilon > 0$, where $n$ is the number of vertices in $G$. We also show that Min Distance $k$-Paired-Dom Set problem is APX-complete for graphs with degree bounded by 3. On the positive side, we present a linear time algorithm to compute the minimum cardinality of a distance $k$-paired-dominating set of a strongly chordal graph $G$ if a strong elimination ordering of $G$ is provided. We show that for a given graph $G$, Min Distance $k$-Paired-Dom Set problem can be approximated with an approximation factor of $1 + \ln 2 + k \cdot \ln(\Delta(G))$, where $\Delta(G)$ denotes the maximum degree of $G$.

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1. Introduction

Suppose $G = (V, E)$ is a simple graph. The open neighborhood of a vertex $v$ in $G$ is the set $N_G(v) = \{u \in V: uv \in E\}$ and the closed neighborhood of $v$ in $G$ is $N_G[v] = \{v\} \cup N_G(v)$. The degree of a vertex $v$ in $G$ is $d_G(v) = |N_G(v)|$. The maximum degree of a graph $G = (V, E)$ is $\Delta(G) = \max\{d_G(v): v \in V\}$. For a subset $S$ of $V$, the subgraph of $G$ induced by $S$ is the graph $G[S]$ with vertex set $S$ and edge set $\{xy \in E: x, y \in S\}$. A matching in $G$ is a set of pairwise nonadjacent edges. For a matching $M$ in $G$, a vertex $v$ is saturated by $M$ if $v$ is incident to some edge of $M$; otherwise $v$ is unsaturated by $M$. A matching $M$ in $G$ is a perfect matching if $G$ has no vertex unsaturated by $M$.

A subset $D \subseteq V$ is a dominating set of $G$ if every vertex in $V \setminus D$ has at least one neighbor in $D$. A subset $D \subseteq V$ is a paired-dominating set of $G$ if $D$ is a dominating set and the subgraph $G[D]$ contains a perfect matching. A paired-dominating set...
set is minimum if the cardinality of $D$ is minimum among all paired-dominating sets of $G$. The concept of paired-domination was introduced by Haynes and Slater [21] and then well studied in the literature [2,4,8–10,12,13,21,28].

The distance between two vertices $x$ and $y$ in $G$ is the minimum length of a path from $x$ to $y$. A subset $D \subseteq V$ is a distance $k$-dominating set of $G$ if every vertex $v$ in $V$ has at least one vertex $u$ in $D$ such that $d_G(u, v) \leq k$. A subset $D \subseteq V$ is a distance $k$-paired-dominating set of $G$ if $D$ is a distance $k$-dominating set and the subgraph $G[D]$ has a perfect matching. The distance $k$-paired-dominating number, denoted by $\gamma_{kP}(G)$, is the minimum cardinality of a distance $k$-paired-dominating set of $G$. The concept of distance $k$-paired-dominination was introduced by Raczek [29] as a generalization of paired-dominination.

Given a graph $G = (V, E)$ and a fixed positive integer $k$, the Min Distance $k$-PAIRED-DOM SET problem is to find a minimum distance $k$-paired-dominating set of $G$. The decision version of Min Distance $k$-PAIRED-DOM SET problem was shown to be NP-complete even restricted to bipartite graphs [29]. Chen et al. [11] presented linear time algorithms to find minimum distance $k$-paired-dominating sets in subclasses of chordal graphs including trees, interval graphs, block graphs, and split graphs etc. However, to the best of our knowledge, no result has been obtained on the approximability of the Min Distance $k$-PAIRED-DOM SET problem.

Strongly chordal graphs is a subclass of chordal graphs and includes directed path graphs, interval graphs, block graphs and trees as subclasses. In this paper, we first study the complexity of Min Distance $k$-PAIRED-DOM SET problem in undirected path graphs and strongly chordal graphs, the two well known subclasses of chordal graphs. Then we concentrate on the approximability of the Min Distance $k$-PAIRED-DOM SET problem. In particular, the main results that will be presented in this paper are summarized as follows:

1. The decision version of the Min Distance $k$-PAIRED-DOM SET problem is NP-complete for undirected path graphs.
2. The minimum cardinality of a distance $k$-paired-dominating set of a given strongly chordal $G = (V, E)$ with a strong elimination ordering can be computed in $O(n + m)$ time, where $n = |V|$ and $m = |E|$.
3. Min Distance $k$-PAIRED-DOM SET problem cannot be approximated within a factor of $1 - \varepsilon$ in $\text{NP}$, where $n$ is the number of vertices in the given graph $G$.
4. Min Distance $k$-PAIRED-DOM SET problem can be approximated with an approximation factor of $1 + \log m$.
5. Min Distance $k$-PAIRED-DOM SET problem is APX-complete for graphs with degree bounded by $3$.

2. Preliminaries

Suppose $G = (V, E)$ is a simple graph. If $d_G(v) = 0$, then $v$ is called an isolated vertex. The open $k$-neighborhood of a vertex $x$ in $G$ is the set $N^k_G(x) = \{y \in V : 1 \leq d_G(x, y) \leq k\}$ and the closed $k$-neighborhood of $x$ in $G$ is $N^k_G(x) = \{y \in V : d_G(x, y) \leq k\}$. An independent set in $G$ is a subset $S \subseteq V$ of pairwise nonadjacent vertices. A clique in $G$ is a subset $C \subseteq V$ of pairwise adjacent vertices. A maximal clique is a clique which is not a proper subset of another clique.

A graph is chordal if every cycle of length at least four has a chord. A vertex $v \in V(G)$ is simplicial if $G[N_G[v]]$ is a clique of $G$. An ordering $(v_1, v_2, \ldots, v_n)$ of $V$ is a perfect elimination ordering (PEO) of $G$ if $v_i$ is a simplicial vertex of $G_i = G[(v_1, v_2, \ldots, v_i)]$ for $1 \leq i \leq n$. It is well known [17] that a graph $G$ is chordal if and only if $G$ has a PEO.

Strongly chordal graphs are introduced by many researchers [7,15,22]. As far as domination problems are concerned, strongly chordal graphs are very important subclass of chordal graphs as the variations of domination problem are efficiently solvable in strongly chordal graphs [6,7,15,22–24,31]. It includes several well known subclasses of chordal graphs such as directed path graphs, interval graphs, block graphs and trees as its subclasses. To define strongly chordal graphs, we refer to the definition given by Farber [15]. A vertex $v$ is simple if the set $\{u \in N_G[v] : u \notin N_G[v]\}$ can be linearly ordered by set inclusion. Alternatively, a vertex $v$ is simple if for any two vertices $x$ and $y$ in $N_G[v]$, either $N_G[x] \subseteq N_G[y]$ or $N_G[y] \subseteq N_G[x]$. An ordering $(v_1, v_2, \ldots, v_n)$ of $V$ is a simple elimination ordering of $G$ if $v_i$ is simple in $G_i = G[(v_1, v_2, \ldots, v_i)]$ for $1 \leq i \leq n$.

**Theorem 2.1** ([15]). A graph is a strongly chordal graph if and only if it admits a simple elimination ordering.

Simple elimination orderings are not the only vertex ordering which characterizes the strongly chordal graphs, as we see in the next theorem. An ordering $\alpha = (v_1, v_2, \ldots, v_n)$ is a strong elimination ordering of $G$ if $N_G[v_i] \subseteq N_G[v_j]$ for $i < j < k$ and $v_i, v_k \notin N_G[v_j]$. Notice that a strong elimination ordering is a perfect elimination ordering and $v_i$ is a simple vertex of the subgraph $G_i$.

**Theorem 2.2** ([15]). A graph is strongly chordal if and only if it admits a strong elimination ordering.

There are many algorithms given by researchers for finding such an ordering. Anstee and Farber [1] presented an $O(n^3)$ time algorithm, Hoffman et al. [22] gave an $O(n^3)$ time algorithm, Lubiw [25] presented an $O(m \log^2 m)$ time algorithm, Paige and Tarjan [26] presented an $O(m \log m)$ time algorithm and Spinrad [30] presented an $O(n^2)$ time algorithm to find a strong elimination ordering of a strongly chordal graph with $n$ vertices and $m$ edges.

Let $F$ be a family of sets. The intersection graph of $F$ is obtained by taking each set in $F$ as a vertex and joining two sets in $F$ if and only if they have a non-empty intersection. Let $C(G)$ be the set of all maximal cliques of a graph $G$ and $C_i(G)$ be the set of all maximal cliques of $G$ containing $v$. Walter [32], Buneman [5] and Gavril [19] have shown that chordal graphs are exactly the intersection graphs of subtrees of a tree. In fact for every chordal graph $G$, there exists a tree $T$ such that
$V(T) = C(G)$ and $T[C_v(G)]$ is a subtree of $T$ for every $v \in V(G)$ such that $G$ is the intersection graphs of the family of paths $\{T[C_v(G)]: v \in V(G)\}$ in $T$. Such a tree is called a clique tree of $G$.

A graph $G$ is called an undirected path graph if $G$ is the intersection graphs of a family of paths of a tree. The following theorem characterizes the undirected path graphs.

**Theorem 2.3** ([20]). A graph $G$ is an undirected path graph if and only if there exists a tree $T$ with $V(T) = C(G)$ such that $T[C_v(G)]$ is a path in $T$ for each $v \in V(G)$.

The following lemma is a result taken from [29] which is a lower bound on $\gamma_p^k(G)$.

**Lemma 2.4** ([29]). If $G = (V, E)$ is a graph without isolated vertices and $\Delta = \Delta(G) \geq 3$, then $\gamma_p^k(G) \geq \frac{\Delta - 2}{\Delta - 1}|V| - 1$.

### 3. Distance $k$-paired-domination in undirected path graphs

The MIN PAIRED-DOM SET problem for a given graph $G$ is to find a minimum paired-dominating set of $G$. For our convenience, we call DECIDE MIN PAIRED-DOM SET problem as the decision version of the MIN PAIRED-DOM SET problem. In this section, we show that the decision version of the MIN DISTANCE $k$-PAIRED-DOM SET problem is NP-complete for undirected path graphs. To do this, we first show that the decision version of MIN DISTANCE 1-PAIRED-DOM SET problem i.e., DECIDE MIN PAIRED-DOM SET problem is NP-complete for undirected path graphs. We do this by providing a polynomial time reduction from the 3-dimensional matching (3-DM) problem which is stated below.

**3-DM Problem.**

**Instance:** A set $M \subseteq W \times X \times Y$, where $W$, $X$ and $Y$ are disjoint sets with $|W| = |X| = |Y| = q$.

**Question:** Does $M$ contain a matching, i.e., a subset $M' \subseteq M$ such that $|M'| = q$ and no two elements of $M'$ agree in any coordinate?

The 3-DM problem is known to be NP-complete [18].

**Theorem 3.1.** The DECIDE MIN PAIRED-DOM SET problem is NP-complete for undirected path graphs.

**Proof.** It is easy to see that the DECIDE MIN PAIRED-DOM SET problem for undirected path graphs is in NP.

We, next, describe a polynomial transformation from 3-DM to DECIDE MIN PAIRED-DOM SET problem in undirected path graphs.

Consider an instance of 3-DM problem consisting of three distinct disjoint sets $W, X$ and $Y$ each of cardinality $q$ and a subset $M = \{m_i = (w_i, x_i, y_i) : w_i \in W, x_i \in X$ and $y_i \in Y$ for $1 \leq i \leq p\}$ of $W \times X \times Y$ having $p$ elements. We construct a tree $T$ having $6p + 3q + 1$ vertices that becomes the clique tree for an undirected path graph $G$. The vertices of the tree are maximal cliques of $G$. The vertex set and edge set of the tree are described below.

For each $1 \leq i \leq p$, let $K_1^i = \{A_i, B_i, C_i, D_i\}$, $K_2^i = \{A_i, B_i, D_i, F_i\}$, $K_3^i = \{C_i, D_i, G_i\}$, $K_4^i = \{A_i, B_i, D_i, E_i\}$, $K_5^i = \{A_i, E_i, H_i\}$ and $K_6^i = \{B_i, E_i, I_i\}$. For each $m_i = (w_i, x_i, y_i) \in M, 1 \leq i \leq p$, we construct a subtree $T_i = (V_i, E_i)$, where $V_i = \{K_1^i : 1 \leq j \leq 6\}$ and $E_i = \{K_3^i K_1^i, K_1^i K_4^i, K_2^i K_4^i, K_4^i K_5^i, K_4^i K_6^i\}$.

Next we construct $L(w_i), P(x_i)$ and $Q(y_i)$ corresponding to each $w_i \in W, x_i \in X$ and $y_i \in Y$ respectively, where $L(w_i) = \{R_i\} \cup \{A_i : w_i \in m_i\}$, $P(x_i) = \{S_i\} \cup \{B_i : x_i \in m_i\}$ and $Q(y_i) = \{T_i\} \cup \{C_i : y_i \in m_i\}$. Let $V_w = \cup_{w_i \in W} L(w_i)$, $V_x = \cup_{x_i \in X} P(x_i)$ and $V_y = \cup_{y_i \in Y} Q(y_i)$.

Let $K = \{A_i, B_i, C_i : 1 \leq i \leq p\}$. Finally, we construct the tree $T$, where $V(T) = \{K\} \cup (\cup_{i=1}^p V_i) \cup (V_w \cup V_x \cup V_y)$ and $E(T) = (\cup_{i=1}^p E_i) \cup (\cup_{i=1}^p [K K_i^i]) \cup [K : v \in V_w \cup V_x \cup V_y]$. The construction of $T$ from an instance of 3-DM is illustrated in Fig. 1.

We then have a graph $G$ with vertex set $\{A_i, B_i, C_i, D_i, E_i, F_i, G_i, H_i, I_i : 1 \leq i \leq p\} \cup \{R_j, S_j, T_j : 1 \leq j \leq q\}$ of size $9p + 3q$, where the undirected path in $T$ corresponding to a vertex $v$ of $G$ consists of those vertices (sets) containing $v$ in the tree $T$. Clearly by **Theorem 2.3**, $G$ is an undirected path graph. It is easy to see that the tree $T$ can be constructed in polynomial time from an instance of 3-DM. Given $T, G$ can also be formed in polynomial time.

**Claim:** $G$ has a paired-dominating set of size $2p + q$ if $q$ is even or $2p + q + 1$ if $q$ is odd if and only if the the answer to the 3-DM problem is yes.
Proof of the Claim. Suppose $D$ is a paired-dominating set of size $2p + q$ if $q$ is even or $2p + q + 1$ if $q$ is odd. Observe that for any $i$, the only way to dominate the vertex set $\{A_i, B_i, C_i, D_i, E_i, F_i, G_i, H_i, I_i\}$ corresponding to $m_i$ with two vertices is to choose $D_i$ and $E_i$, and that any larger paired-dominating set might just as well consist of $A_i, B_i, C_i$, since none of the other possible vertices dominate any vertex outside of the set. Let $q$ is even. Then each $A_i$ can be paired with $B_i$ and $C_i$ can be paired with a $C_j$ where $i \neq j$. Consequently, $D$ consists of $A_i, B_i, C_i$ for $t$ number of $m_i$'s; and $D_i$ and $E_i$ for $p - t$ number of other $m_i$'s, and at least \( \max(3(q-t),0) \) number of $R_i, S_i, T_i$. Then

\[
2p + q = |D| \geq 3t + 2(p-t) + 3(q-t) = 2p + 3q - 2t
\]

and so $t \geq q$. Now suppose $q$ is odd. Then each $A_i$ can be paired with $B_i$ and $C_i$ can be paired with $C_j, i \neq j$ leaving one $C_i$. So, for this particular $C_i, D$ must contain one of $G_i$ or $D_i$. Consequently, $D$ consists of $A_i, B_i, C_i$ for $t$ number of $m_i$'s; and $D_i$ and $E_i$ for $p - t$ number of other $m_i$'s, and at least \( \max(3(q-t),0) \) number of $R_i, S_i, T_i$. Then

\[
2p + q + 1 = |D| \geq 3t + 2(p-t) + 3(q-t) = 2p + 3q - 2t
\]

and so $t \geq q + 1$. Picking the $q$ triples $m_i$, for which $A_i, B_i$ and $C_i$ are in $D$ form a matching $M'$ of size $q$.

Conversely, suppose the answer to the 3-DM problem is yes. So there is a matching $M'$ of size $q$. Without loss of generality, let $M' = \{m_1, m_2, \ldots, m_q\}$. If $q$ is even, let $D = \{A_i, B_i, C_i; m_i \in M'\} \cup \{D_i, E_i; m_i \in M \setminus M'\}$ and if $q$ is odd, let

\[
D = \{A_i, B_i, C_i; m_i \in M' \setminus \{m_1\}\} \cup \{A_1, B_1, C_1, D_1; m_1 \in M'\} \cup \{D_i, E_i; m_i \in M \setminus M'\}.
\]

It is straightforward to check that $D$ is a paired-dominating set of $G$ of size $3q + 2(p - q) = 2p + q$ if $q$ is even and $3(q - 1) + 2(p - q) + 4 = 2p + q + 1$ if $q$ is odd. \(\square\)

Hence the **Decide Min Paired-Dom Set** problem for undirected path graphs is NP-complete. \(\square\)
Now suppose $G = (V, E)$ is an undirected path graph. Given a positive integer $k$, let $G_k = (V_k, E_k)$, where $V_k = V \cup \{v_i: v \in V$ and $i = 1, 2, \ldots, k - 1\}$ and $E_k = E \cup \{v_{i-1}v_i: v \in V$ and $i = 1, 2, \ldots, k - 1\}$, where $v_0 = v$. Since $G$ is an undirected path graph, there is a tree $T$ and a set of paths $\{P_v: v \in V\}$ of $T$ such that $uv \in E$ if and only if $P_u \cap P_v \neq \emptyset$. For each path $P_v$, consider an end point $v^*$ of $P_v$. Construct a tree $T_v$ which results from $T$ by attaching a new path $v^* = v_0, v_1, v_2, \ldots, v_{k-1}$ of length $k - 1$ at $v^*$ for each $v \in V$. Then it is clear that $G_k$ is the intersection graph of the set of paths $\bigcup_{v \in V} \{P_v \cup \{v_0v_1, v_1v_2, \ldots, v_{k-2}v_{k-1}\}\}$. Hence $G_k$ is an undirected path graph.

We denote DECIDE MIN DISTANCE $k$-PAIRED-DOM SET problem as the decision version of MIN DISTANCE $k$-PAIRED-DOM SET problem. Since DECIDE MIN PAIRED-DOM SET problem is NP-complete, we can have the following theorem.

**Theorem 3.2.** For any fixed positive integer $k$, the DECIDE MIN DISTANCE $k$-PAIRED-DOM SET problem is NP-complete for undirected path graphs.

### 4. Distance $k$-paired-domination in strongly chordal graphs

There is a drawback for the definition of distance $k$-domination. Namely, a graph with isolated vertices has no distance $k$-dominating sets, although a graph without isolated vertices always has a distance $k$-dominating set, such as the vertex set of a maximum matching.

For this and the purpose below, we define an equivalent way for which also covers the case of graphs containing isolated vertices. For a subset $S \subseteq V(G)$, the paired-weight of $S$ is $\omega_p(S) = |S| - 2|M|$, where $M$ is a maximum matching of $G[S]$. Notice that if $G[S]$ has a perfect matching, then $\omega_p(S) = |S|$. Let $\gamma_{lp}^k(G)$ be the minimum paired-weight $\omega_p(D)$ of a distance $k$-dominating set $D$ of $G$.

**Lemma 4.1.** If $G$ is a graph without isolated vertices, then $\gamma_{lp}^k(G) = \gamma_{lp}^k(G)$.

**Proof.** Suppose $D$ is a distance $k$-dominating set with $\omega_p(D) = \gamma_{lp}^k(G)$. Let $M$ be a maximum matching of $G[D]$, and $U = \{u \in D|u \text{ is not matched by } M\}$. Consider the set system $\{A_u: u \in U\}$, where $A_u = \{v \notin D: \delta_c(v, u) \leq k\}$. As $G$ has no isolated vertex, each $A_u \neq \emptyset$. We claim that $\bigcup_{u \in U} A_u \neq \emptyset$ for any subset $I \subseteq U$. Suppose to the contrary that $\bigcup_{u \in U} A_u < |I|$ for some subset $I \subseteq U$. Then $(D \setminus I) \cup \bigcup_{u \in U \setminus I} A_u$ is a distance $k$-dominating set of paired-weight smaller than $D$, which is impossible. By Hall’s theorem, $\{A_u: u \in U\}$ has an SDR $\{u^*: u \in U\}$. Then $D^* = D \cup \{u^*: u \in U\}$ gives a distance $k$-dominating set of size $|D^*| = |D| + |U| = \omega_p(D)$. This gives that $\gamma_{lp}^k(G) \leq \gamma_{lp}^k(G)$.

On the other hand, since a distance $k$-dominating set $D$ of $G$ has paired-weight $\omega_p(D) = |D|$, we have $\gamma_{lp}^k(G) \geq \gamma_{lp}^k(G)$. Hence $\gamma_{lp}^k(G) = \gamma_{lp}^k(G)$. \Box

To give a good algorithm for the problem of distance $k$-domination in strongly chordal graphs, we design the algorithm in a more general setting as follows. For every vertex $v$ of a graph $G$ there is a labeling $L(v) = (a_v, b_v)$, where $a_v$ is a nonnegative integer and $b_v$ a positive integer. An $L$-dominating set of $G$ is a subset $D \subseteq V(G)$ such that for any vertex $v \in V(G)$ either $\delta(v, u) \leq a_v$ for some $u \in D$ or $\delta(v, u) + b_v \leq a_v$ for some $u \in V\setminus D$. Suppose for each vertex $v \in V(G)$, we have another label $c_v \in \{0, 2\}$. The $c$-paired-weight of a subset $S \subseteq V(G)$ is $\omega_{cp}(S) = 2|S| - 2|M|$, where $M$ is a maximum matching of the subgraph $G[v \in S: c_v = 0]$. The $c$-L-PAIRED-DOMINATION NUMBER $\gamma_{lp}(G)$ of $G$ is the minimum $c$-paired-weight $\omega_{cp}(D)$ of an $L$-dominating set $D$ of $G$. For the case when $(a_v, b_v, c_v) = (k, k + 1, 0)$ for all $v \in V(G)$, $\gamma_{lp}(G) = \gamma_{lp}^k(G)$.

**Lemma 4.2.** Suppose $G$ is a strong chordal graph with a strong elimination ordering $(v_1, v_2, \ldots, v_n)$ and $G_i = G[v_1, v_{i+1}, \ldots, v_n]$ for $1 \leq i \leq n$. Let $c$ be the largest index such that $v_c \in N_{G_n}(v_i)$. If $a_{v_i} > a_{v_i} > 0$, and $b_{v_i} + 1 > a_{v_i}$ and $a_{v_i} > 0$ for $v_i \in N_{G_n}(v_i)$, then $\gamma_{lp}(G) = \gamma_{lp}(G_{i+1})$ where $(c', L')$ is the same as $(c, L)$ except that $a_{v_i}' = \min\{a_{v_i}, a_{v_i} - 1\}$ and $b_{v_i}' = \min\{b_{v_i}, b_{v_i} + 1\}$ for $v_i \in N_{G_n}(v_i)$.

**Proof.** Notice that $G_{i+1}$ is a distance invariant subgraph of $G_i$. So we use $d(v, u)$ for the distance between $v$ and $u$ in all $G_i$. Suppose $D'$ is an $L'$-dominating set of $G_{i+1}$ with $\omega_{cp}(D') = \gamma_{lp}(G_{i+1})$. Since $(c', L')$ is the same as $(c, L)$ except that $a_{v_i}' = \min\{a_{v_i}, a_{v_i} - 1\} \leq a_{v_i}$, conditions on $L'$-domination implies conditions on $L$-domination for all vertices in $V(G_{i+1})$. Conditions on $L'$-domination for $v_i$ imply that either $d(v, v_i) \leq 1 + d(v, v_i) \leq 1 + a_{v_i}' < a_{v_i}$ for some $v_i \in V(G_{i+1}) \cap D'$ or else $d(v, v_i) + b_{v_i} \leq 1 + d(v, v_i) + b_{v_i}' \leq 1 + a_{v_i}' \leq a_{v_i}$ for some $v_i \in V(G_{i+1})$. This gives conditions on $L'$-domination for $v_i$. Hence, $D'$ is an $L$-dominating set of $G_i$. As $c' = c, \gamma_{lp}(G_i) \leq \omega_{cp}(D') = \omega_{cp}(D') = \gamma_{lp}(G_{i+1})$.

On the other hand, suppose $D$ is an $L$-dominating set of $G_i$ with $\omega_{cp}(D) = \gamma_{lp}(G_i)$. We consider two cases.

**Case 1.** $D \subseteq V(G_{i+1})$.

In this case, Conditions on $L'$-domination for $v_i \in V(G_{i+1}) \setminus \{v_i\}$ give that either $d(v_i, v_i) \leq a_{v_i}$ for some $v_i \in D$ or else $d(v_i, v_i) + b_{v_i} \leq a_{v_i}$ for some $v_i \in V(G_i)$. For the case if $d(v_i, v_i) + b_{v_i} \leq a_{v_i}$ then $d(v_i, v_i) + b_{v_i}' \leq d(v_i, v_i) + b_{v_i} \leq a_{v_i}$, where $v_i \in N_{G_n}(v_i)$. This implies that $d(v_i, v_i) + b_{v_i}' \leq a_{v_i}$. This gives the conditions on $L'$-domination on $G_{i+1}$. Hence $D$ is also a $L'$-dominating set of $G_{i+1}$. Since $c' = c, \gamma_{lp}(G_{i+1}) \leq \omega_{cp}(D) = \omega_{cp}(D) = \gamma_{lp}(G_i)$.

**Case 2.** $v_i \notin D$.

If $d(v_i, v_i) \leq a_{v_i}$, then $d(v_i, v_i) + 1 \leq a_{v_i}$, where $v_i \in N_{G_n}(v_i)$ and hence $d(v_i, v_i) \leq a_{v_i} - 1 \leq a_{v_i}$. If $d(v_i, v_i) + b_{v_i} \leq a_{v_i}$, then $d(v_i, v_i) + b_{v_i}' \leq d(v_i, v_i) + b_{v_i} \leq d(v_i, v_i) + b_{v_i} \leq d(v_i, v_i) + 1 + b_{v_i} \leq a_{v_i}$ since $d(v_i, v_i) + b_{v_i} \leq d(v_i, v_i) + 1 + b_{v_i} \leq a_{v_i}$.
Therefore, $D' = (D \setminus \{v_i\}) \cup \{v_f\}$ is also a $L$-dominating set of $G_i$. Case 1 confirms that $D'$ is also $L'$-dominating set of $G_{i+1}$ and $\gamma_{L'}(G_{i+1}) \geq \gamma_{L}(G_i)$. Hence $\gamma_{L}(G_i) = \gamma_{L'}(G_{i+1})$.

**Lemma 4.3.** Suppose $G$ is a strong chordal graph with a strong elimination ordering $(v_1, v_2, \ldots, v_n)$ and $G_i = G_i[v_1, v_{i+1}, \ldots, v_n]$, $1 \leq i \leq n$, then $\gamma_{L}(G_i) = \gamma_{L'}(G_{i+1})$ where $(c', L')$ is the same as $(c, L)$ except that $b'_{v_i} = \min\{b_{v_i} + 1\}$ for $v_i \in N_{G_i}(v_i)$.

**Proof.** Notice that $G_{i+1}$ is a distance invariant subgraph of $G_i$. So we use $d(v, w)$ for the distance between $v$ and $w$ in all $G_i$. Suppose that $D'$ is an $L'$-dominating set of $G_{i+1}$ with $w_d(D') = \gamma_{L'}(G_{i+1})$. If for $v_i \in V(G_{i+1})$, there is a vertex $v \in D'$ such that $d(v, v_i) \leq a'_p$, then $d(v, v_i) \leq a_p$. So we need only to consider the case when for $v_i \in V(G_{i+1})$, $d(v_i, v_i) + b'_{v_i} \leq a_{v_i}$ for some $v_i \in N_{G_i}(v_i)$. If $d(v_i, v_i) + b'_{v_i} = a_{v_i}$, then we are done. Otherwise, since $d(v_i, v_i) = d(v_i, v_i) + 1$, we get $d(v_i, v_i) + b'_{v_i} = d(v_i, v_i) + 1 + b_{v_i} = d(v_i, v_i) + b'_{v_i} \leq a_{v_i}$. Hence $D'$ is also an $L$-dominating set of $G_i$. As $c' = c$, we have $\gamma_{L}(G_i) \leq w_{cp}(D') = w_{cp}(D') = \gamma_{L'}(G_{i+1})$.

On the other hand, suppose that $D$ is an $L$-dominating set of $G_i$ with $w_{cp}(D) = \gamma_{L}(G_i)$. We need to consider two cases:

**Case 1.** $v_i \notin D$.

In this case, conditions on $L$-domination for $v_i \in V(G_{i+1}) \setminus \{v_i\}$ give that either $d(v_i, v_i) \leq a_{v_i} = a'_{v_i}$ for some $v_i \in D$ or $d(v_i, v_i) + b_{v_i} \leq a_{v_i}$ for some $v_i \in V(G_{i+1})$. For the latter case with $v_i = v_i$, we have $d(v_i, v_i) + b'_{v_i} \leq a_{v_i}$. Since $c' = c$, $\gamma_{L'}(G_{i+1}) \leq w_{cp}(D) = w_{cp}(D) = \gamma_{L}(G_i)$.

**Case 2.** $v_i \in D$.

If $d(v_i, v_i) \leq a_{v_i}$, then $d(v_i, v_i) + b_{v_i} \leq a_{v_i}$, where $v_i \in N_{G_i}(v_i)$ and hence $d(v_i, v_i) \leq a_{v_i} - 1 \leq a_{v_i}$. If $d(v_i, v_i) + b_{v_i} \leq a_{v_i}$, then $d(v_i, v_i) + b'_{v_i} \leq d(v_i, v_i) + 1 + b_{v_i} = d(v_i, v_i) + b_{v_i} \leq a_{v_i}$. Hence $D'$ is also an $L'$-dominating set of $G_{i+1}$ and $\gamma_{L'}(G_{i+1}) \leq \gamma_{L}(G_i)$.

Hence $\gamma_{L}(G_i) = \gamma_{L'}(G_{i+1})$. □

**Lemma 4.4.** Suppose $G$ is a strong chordal graph with a strong elimination ordering $(v_1, v_2, \ldots, v_n)$ and $G_i = G_i[v_1, v_{i+1}, \ldots, v_n]$, for $1 \leq i \leq n$. If $a_{v_i} = 0$ and $c_{v_i} = 2$, then $\gamma_{L}(G_i) = \gamma_{L'}(G_i) + 2$, where $(c', L')$ is the same as $(c, L)$ except that $c'_v = 0$.

**Proof.** The proof is easy. Any $cL$-dominating set $D$ of $G_i$ can be converted to an $c'L'$-dominating set of $G_i$ by setting $c'_{v_i} = c_{v_i} - 2$. So $\gamma_{L'}(G_i) \leq w_{cp}(D) = \gamma_{L}(G_i) - 2$. Similarly any $c'L'$-dominating set of $G_i$ can be converted to an $cL$-dominating set $D'$ of $G_i$ by setting $c'_{v_i} = c'_{v_i} + 2$. So $\gamma_{L}(G_i) = w_{cp}(D) + 2 \leq \gamma_{L'}(G_i) + 2$ implying the equality. □

**Lemma 4.5.** Suppose $G$ is a strong chordal graph with a strong elimination ordering $(v_1, v_2, \ldots, v_n)$ and $G_i = G_i[v_1, v_{i+1}, \ldots, v_n]$, for $1 \leq i \leq n$. If $a_{v_i} = 0$ and $c_{v_i} = 0$, then $D$ is a minimum $cL$-dominating set of $G_i$ if and only if $D' \cup \{v_i\}$, where $D'$ is a minimum $c'L'$-dominating set of $G_{i+1}$ where $(c', L')$ is same as $(c, L)$ except that $b'_{v_i} = 1$ for $v_i \in N_{G_i}(v_i)$.

**Proof.** Notice that $G_{i+1}$ is a distance invariant subgraph of $G_i$. So we use $d(v, w)$ for the distance between $v$ and $w$ in all $G_i$. Since $a_{v_i} = 0$, $v_i$ is not $L$-dominated by any other vertex of $G_i$. Hence $v_i$ must be in any $L$-dominating set of $G_i$, in particular, $v_i \in D$. Let $D' = D \setminus \{v_i\}$. If $D'$ is not an $L'$-dominating set of $G_{i+1}$, then there exists $v_{k} \in V(G_{i+1})$ such that $d(v_{k}, D') > a_{v_{k}}$ and $d(v_{k}, v_{i}) + b_{v_{i}} > a_{v_{i}}$ for any vertex $v_{i}$. Thus either $d(v_{k}, v_{i}) \leq a_{v_{i}}$ or $d(v_{k}, v_{i}) + b_{v_{i}} \leq a_{v_{i}}$ for some $v_{i}$. If $d(v_{k}, v_{i}) \leq a_{v_{i}}$, then let $w \in N_{G_i}(v_i)$ be the vertex on the shortest path between $v_i$ and $v_k$. Then $d(v_k, v_k) + b'(w) = d(v_k, w) + d(w, v_i) = d(v_k, v_i) - a_{v_i}$, which is a contradiction.

Next consider the latter case i.e. $d(v_k, x) + b_x \leq a_{v_x}$ for $v_x \in G_{i+1}$. Either $x \in N_{G_i}(v_i)$ or $x = v_i$. If $x \notin N_{G_i}(v_i)$, then $b'_{v_i} = 1$ and $d(v_k, x) + b_x \leq a_{v_x}$, which is a contradiction. If $x = v_i$, then $d(v_k, v_i) + b_{v_i} \leq a_{v_i}$. Moreover, $d(v_k, w) + b'_{v_i} \leq a_{v_i}$, where $w \in N_{G_i}(v_i)$ is a vertex on the shortest path between $v_i$ and $v_k$. Thus, $D'$ is an $L'$-dominating set of $G_{i+1}$. As $c' = c$, $\gamma_{L'}(G_{i+1}) \leq w_{cp}(D') = w_{cp}(D) - c_{v_i} = \gamma_{L}(G_i)$.

Conversely, let $a'_{v_i} = a_{v_i}$ for all $v_{i} \in V(G_{i+1})$. Therefore, if $d(v_i, v_i) \leq a'_{v_i}$ in $G_{i+1}$, then $d(v_i, v_i) \leq a_{v_i}$ in $G_i$. Let $v_i \in V(G_{i+1})$ such that $d(v_i, v_i) + b'_{v_i} \leq a_{v_i}$. If $v_i \notin N_{G_i}(v_i)$, then $d(v_i, v_i) + b_{v_i} \leq a_{v_i}$. If $v_i \in N_{G_i}(v_i)$, then $d(v_i, v_i) + b_{v_i} = d(v_i, v_i) + 1 = d(v_i, v_i) + d(v_i, v_i) = d(v_i, v_i) \leq a_{v_i}$. Therefore, $D' \cup \{v_i\}$ is an $cL$-dominating set of $G_i$. As $c' = c$, we have $\gamma_{L}(G_i) \leq w_{cp}(D') + c_{v_i} = w_{cp}(D') + c_{v_i} = \gamma_{L'}(G_{i+1})$. □

The proof of the following lemma is immediate and hence is omitted.

**Lemma 4.6.** If $v_i$ is an isolated vertex of a graph $G$ and $D$ be a minimum $L$-dominating set of $G$, then $v_i \in D$ if and only if $a_{v_i} < b_{v_i}$.

Depending on the above lemmas, we now present an algorithm that computes a $L$-dominating set of $G$ such that $w_{D}(G) = \gamma_{L}(G)$.

Notice that in Lines 16–23, we settle the vertex $v_i$ to be dominated by $D$ such that $w_{D}(G)$ will be minimum. If $d(v_i, v) \geq k + 2$, then it is better to include the vertex $v_i$ in $D$. But it happens that if $d(v_i, v) \geq k + 1$ for some vertex $v \in D$, then a
vertex $v^* \in N_G(v)$ can be included by making $c_{v^*} = 0$ so that $d(v_n, v^*) \leq k$ and $w_{cp}(D)$ can be minimum. To do so, we take a BFS tree rooted at $v_n$ and check in the $(k + 1)$-th level whether there is a vertex $v$ which is also in $D$. If we find such a vertex, then we look for a parent of $v$. Note that such a vertex exists. All these can be done in at most $O(n + m)$ time.

It is obvious that the running time of the algorithm Algorithm 1 is at most $(n + m)$ time, where $n = |V|$ and $m = |E|$. Now after getting the set $D$, we need to find $w_{cp}(D) = 2|D| - 2|M|$, where $M$ is a maximum matching in $G[D]$ as $c_v = 0$ for each $v \in D$. Note that $G[D]$ is a strongly chordal graph. To apply the algorithm for maximum matching presented in [14], we require a strong elimination ordering of $G[D]$ which requires $O(n^2)$ time for computation. But we do it in $O(n + m)$ time in the following way:

We first label all the vertices of $D$ as ‘Red’ and scan the vertices of $G$ in the strong elimination ordering $\sigma = (v_1, v_2, \ldots, v_n)$ of $G$. Whenever we encounter a vertex $v_i$ with Red label, we look for the smallest indexed neighbor $v_j$ ($j > i$) of $v_i$ and change the label of $v_i$ and $v_j$ as ‘Black’. The set of edges whose end points are marked as Black form a maximum matching of $G[D]$. If this is not true, then there exists a maximum matching of $G[D]$, say $M'$ such that $v_iv_j \notin M'$. Then we need to consider two cases.

**Case 1.** $v_jv_k \in M'$ for some $k > j$ and $v_k$ is labeled as Red.

In this case if $v_jv_r \in M'$, then $v_jv_k \notin E$. So $M'' = (M' \setminus \{v_jv_k, v_jv_r\}) \cup \{v_jv_r, v_rv_k\}$ is a required maximum matching of $G[D]$. If $v_jv_k$ is not a matched vertex with regard to $M'$, then $M'' = (M' \setminus \{v_jv_k\}) \cup \{v_jv_r\}$ is a required maximum matching of $G[D].$

**Case 2.** $v_j$ is not a matched vertex with regard to $M'$.

Then $v_jv_r \in E$ for some Red labeled vertex $v_r$. Then $M'' = (M' \setminus \{v_jv_r\}) \cup \{v_jv_r\}$ is a required maximum matching of $G[D].$

Therefore, the distance $k$-paired-domination number of a strongly chordal graph can be found in $O(n + m)$ time if a strong elimination ordering of $G$ is provided.

**Theorem 4.7.** The distance $k$-paired-domination number of a strongly chordal graph $G = (V, E)$ can be found in $O(n + m)$ time if a strong elimination ordering of $G$ is provided, where $n = |V|$ and $m = |E|$.

5. **Inapproximability of Min Distance $k$-Paired-Dom Set problem**

Given a nonempty set $\mathcal{U}$ and a family $\delta$ of subsets of $\mathcal{U}$, the Min Set Cover problem is to find a minimum cardinality set $S \subseteq \delta$ such that every $u \in \mathcal{U}$ belongs to some $S' \in S$. In this section, we present the inapproximability of Min Distance $k$-Paired-Dom Set problem within $O(\log n)$ factor. For this we need the following result about Min Set Cover problem.
Theorem 5.1 ([16]). Unless \( \text{NP} \subseteq \text{DTIME}(n^{O(\log \log n)}) \), MIN SET COVER cannot be approximated within a factor of \((1 - \varepsilon) \ln n\), for any \( \varepsilon > 0 \), where \( n = |\mathcal{U}| \).

Theorem 5.2. Let \( G = (V, E) \) be a graph with \( n \) vertices and without isolated vertices. Unless \( \text{NP} \subseteq \text{DTIME}(n^{O(\log \log n)}) \), MIN DISTANCE \( k \)-PAIRED-DOM SET cannot be approximated within a factor of \((1 - \varepsilon) \ln n\), for any \( \varepsilon > 0 \).

Proof. Given SC = \( (\mathcal{U}, \mathcal{S}) \), an instance of the MIN SET COVER problem, we give an approximation preserving reduction to MIN DISTANCE \( k \)-PAIRED-DOM SET problem as follows:

Let \( |\mathcal{U}| = r \) and \( |\mathcal{S}| = s \). We construct a graph \( G = (V, E) \) as follows:

- For each \( u_i \in \mathcal{U} \), \( 1 \leq i \leq r \), introduce two vertices \( u_i^1 \) and \( u_i^2 \).
- For each \( S_j \in \mathcal{S} \), \( 1 \leq j \leq s \), introduce two vertices \( S_j^1 \) and \( S_j^2 \).
- If \( u_i \in S_j \), introduce the edges \( u_i^1 S_j^1 \) and \( u_i^2 S_j^2 \).
- We introduce all possible edges to make \( \{S_1^1, S_1^2, \ldots, S_s^1, S_s^2\} \) a clique.
- For each \( u_i^1 \), \( 1 \leq i \leq r \), we attach a path \( u_i^1 - v_i - \cdots - u_i^1 \) to \( u_i^1 \).
- For each \( u_i^2 \), \( 1 \leq i \leq r \), we attach a path \( u_i^2 - w_i - \cdots - w_i^2 \) to \( u_i^2 \).

The construction of \( G \) is illustrated in Fig. 2.

Let \( S^* \subseteq \mathcal{S} \) be an optimal set cover of \( \mathcal{U} \). Then it is clear that \( \{S_1^1, S_1^2, \ldots, S_s^1, S_s^2\} \) is a distance \( k \)-paired-dominating set of \( G \). So \( \gamma^*_k(G) \leq 2|S^*| \).

On the other hand, assume that \( kPD^* \) be a minimum distance \( k \)-paired-dominating set of \( G \). Without loss of generality, we can assume that \( kPD^* \subseteq \{u_1^1, u_1^2, \ldots, u_r^1, u_r^2, S_1^1, \ldots, S_s^1, S_s^2\} \). Let \( D_p = kPD^* \cap \{S_1^1, S_1^2, \ldots, S_s^1, S_s^2\} \) for \( p = 1, 2 \). Without loss of generality, we assume that \( |D_1| \leq |D_2| \). Let \( S' = \{S_j : S_j^1 \in D_1\} \). Then \( |S'| \leq \frac{|kPD^*|}{2} \). Since \( \{u_1^1, u_1^2, u_2^1, \ldots, u_r^1\} \) is an independent set, \( S' \) is a set cover of \( \mathcal{U} \). Hence \( |S^*| \leq |S'| \leq \frac{|kPD^*|}{2} \), where \( S^* \) is an optimal solution to SC. So we have \( \gamma^*_k(G) \leq 2|S^*| \).

Given any solution \( S' \) of SC, \( kPD = \{S_1^1, S_1^2, \ldots, S_s^1, S_s^2\} \) is a distance \( k \)-paired-dominating set of \( G \). So given a set cover \( S' \) of \( \mathcal{U} \), one can find a distance \( k \)-paired-dominating set \( kPD \) of \( G \) with \( |kPD| = 2|S'| \). Now \( \frac{|kPD|}{|kPD^*|} = \frac{|S'|}{|S^*|} = \frac{|S_1^1|}{|S^*_1|} \).

Now suppose that there exists a polynomial time algorithm to approximate MIN DISTANCE \( k \)-PAIRED-DOM SET problem within a factor of \((1 - \varepsilon) \ln N\) for graphs with \( N \) vertices. Since \( N = 2(|\mathcal{U}| + |\mathcal{S}|) + 2(k - 1)|\mathcal{U}| \), we have \( N \leq 4n + 2(k - 1)n = 2(k + 1)n \). Then

\[
\frac{|S'|}{|S^*|} = \frac{|kPD|}{|kPD^*|} \leq (1 - \varepsilon) \ln(2(k + 1)n) = (1 - \varepsilon)(\ln(2k + 2) \ln n) = (1 - \varepsilon) \ln n \left( 1 + \frac{\ln(2k + 2)}{\ln n} \right).
\]

For sufficiently large \( n \), the term \( 1 + \frac{\ln(2k + 2)}{\ln n} \) can be bounded by \( 1 + \frac{\varepsilon}{10} \). Now we have

\[
(1 - \varepsilon) \ln n \left( 1 + \frac{\ln(2k + 2)}{\ln n} \right) \leq (1 - \varepsilon') \ln n,
\]

where \( \varepsilon' \leq \frac{\varepsilon}{10} \). This contradicts the Theorem 5.1 and hence the result follows.

Notice that the constructed graph \( G \) is a chordal graph. Again if we make the graph \( G([S_1^1, S_1^2, \ldots, S_s^1, S_s^2]) \) a complete bipartite graph, then \( G \) is a bipartite graph and the proof goes through. So we have the following corollary.

Corollary 5.3. Let \( G = (V, E) \) be a chordal graph (bipartite graph) with \( n \) vertices and without isolated vertices. Unless \( \text{NP} \subseteq \text{DTIME}(n^{O(\log \log n)}) \), MIN DISTANCE \( k \)-PAIRED-DOM SET cannot be approximated within a factor of \((1 - \varepsilon) \ln n\), for any \( \varepsilon > 0 \).
6. Approximation algorithm for \textbf{Min Distance }$k$-Paired-Dom Set problem

In this section, we present an approximation algorithm, namely APP\-ROX-$k$-PDS that computes a distance $k$-paired-dominating set of a given graph $G = (V, E)$. The approximation ratio of the algorithm APP\-ROX-$k$-PDS is $1 + \ln(2 + k \cdot \ln(\Delta(G)))$, where $\Delta(G)$ is the maximum degree of $G$.

Algorithm 2: APP\-ROX-$k$-PDS

\begin{itemize}
\item \textbf{Input}: A graph $G = (V, E)$ without isolated vertices and a positive integer $k$.
\item \textbf{Output}: A distance $k$-paired-dominating set $kPD$ of $G$.
\end{itemize}

\begin{algorithmic}
1. $kPD = \emptyset$;
2. $i = 0, L_i = \emptyset$;
3. \textbf{while} $(V \setminus (L_0 \cup \cdots \cup L_{i-1}) \neq \emptyset)$ \textbf{do}
4. \hspace{1em} $i = i + 1$;
5. \hspace{1em} Choose two vertices $u, v \in V \setminus kPD$ such that $uv \in E$ and $|N^k_G(u) \cup N^k_G(v)) \setminus (L_0 \cup \cdots \cup L_{i-1})|$ is maximized;
6. \hspace{1em} $L_i = N^k_G(u) \cup N^k_G(v)$;
7. \hspace{1em} $L_i = L_i \setminus (L_0 \cup \cdots \cup L_{i-1})$;
8. \hspace{1em} $kPD = kPD \cup \{u, v\}$;
\end{algorithmic}

\begin{lemma}
Given a graph $G = (V, E)$ and an integer $k > 0$, the set $kPD$ computed by APP\-ROX-$k$-PDS is a distance $k$-paired-dominating set of $G$.
\end{lemma}

\begin{proof}
At each step, two adjacent vertices are computed by APP\-ROX-$k$-PDS. Since a graph $G = (V, E)$ without isolated vertices possesses a distance $k$-paired-dominating set. The set $kPD$ computed by APP\-ROX-$k$-PDS must be a distance $k$-paired-dominating set of $G$. \hfill $\square$
\end{proof}

\begin{lemma}
For each $u \in V$, there exists exactly one set $L_i$ that contains $u$.
\end{lemma}

\begin{proof}
By Lemma 6.1, for each vertex $u \in V$, there exists a vertex $v \in kPD$ such that $d_G(u, v) \leq k$. Thus $u$ is contained in at least one set $L_i$. When $u$ is first contained in the set $L_0$, then $u \in L_0$ for any $r > i_0$. For any $r > i_0$, $u \notin L_i$ as $u \in L_0 \cup \cdots \cup L_r$.
\end{proof}

\begin{theorem}
Let $G = (V, E)$ be a graph of maximum degree $\Delta(G)$ without isolated vertices and $k$ be a positive integer. Then \textbf{Min Distance }$k$-Paired-Dom Set in $G$ can be approximated with an approximation ratio of $1 + \ln(2 + k \cdot \ln(\Delta(G)))$.
\end{theorem}

\begin{proof}
For any set $Y \neq \emptyset$, we have $\sum_{i \in Y} \frac{1}{|i|} = 1$. Therefore, we have

$$|kPD| = 2 \sum_{i=1}^{\lceil |kPD| \rceil} \sum_{u \in L_i} \frac{1}{|L_i|}. $$

By Lemma 6.2, there exists exactly one index $i \in \{1, 2, \ldots, \lceil |kPD| \rceil \}$ such that $u \in L_i$ for each $u \in V$. Now let $c_u = \frac{1}{|L_i|}$. So, we have

$$|kPD| = 2 \sum_{i=1}^{\lceil |kPD| \rceil} \sum_{u \in L_i} \frac{1}{|L_i|} = 2 \sum_{u \in V} c_u. $$

Let $kPD^*$ be a minimum distance $k$-paired-dominating set of $G$ and $M$ be a perfect matching in $G[kPD^*]$. Thus, for each vertex $w \in V$, there is at least one vertex $v$ in $kPD^*$ such that $d_G(u, v) \leq k$. In other words, there exist $u, v \in kPD^*$ such that $uv \in E$ and $w \in N^k_G(u) \cup N^k_G(v)$. Therefore, we have

$$\sum_{w \in V} c_w \leq \sum_{w \in M} \sum_{w \in N^k_G(u) \cup N^k_G(v)} c_w. $$

Let $uv \in M$ and $z_i = |(N^k_G(u) \cup N^k_G(v)) \setminus (L_0 \cup \cdots \cup L_i)|$ for $i = 0, 1, 2, \ldots, \lceil |kPD| \rceil$. Notice that $z_i \geq z_j$ for $i = 1, 2, \ldots, \lceil |kPD| \rceil$. Suppose that $\ell$ is the smallest index such that $z_\ell = 0$. At the $i$-th step of APP\-ROX-$k$-PDS, $L_i$ contains $z_i$ vertices in $N^k_G(u) \cup N^k_G(v)$. So, we have

$$\sum_{w \in N^k_G(u) \cup N^k_G(v)} c_w = \sum_{i=1}^\ell \frac{(z_{i-1} - z_i)}{|L_i|}. $$
Note that the set $I_i$ is chosen such that $|(N_G^k(u) \cup N_G^k(v)) \setminus (L_0' \cup \cdots \cup L_{i-1}')| = z_i$. Hence, it follows that
\[
\sum_{u \in N_G^k(u) \cup N_G^k(v)} c_u \leq \sum_{i=1}^{l} (z_{i-1} - z_i) \frac{1}{z_{i-1}}.
\]
For all integers $a < b$, we know that $H(b) - H(a) \geq \frac{b-a}{b}$, where $H(p) = \sum_{i=1}^{l} \frac{1}{z_{i-1}}$ with $H(0) = 0$. Thus,
\[
\sum_{u \in N_G^k(u) \cup N_G^k(v)} c_u \leq \sum_{i=1}^{l} (H(z_{i-1}) - H(z_i)) = H(|N_G^k(u) \cup N_G^k(v)|).
\]
Since the maximum degree of $G$ is $\Delta(G)$, we have $|N_G^k(u) \cup N_G^k(v)| \leq 2 \cdot (\Delta(G))^k$ for each $uv \in M$. So it follows that
\[
|kPD| \leq 2 \sum_{u \in V} c_u \leq 2 \sum_{u \in M \setminus N_G^k(u) \cup N_G^k(v)} \sum_{v \in M} c_w \leq 2 \sum_{u \in M} H(|N_G^k(u) \cup N_G^k(v)|) \leq 2 \sum_{u \in M} H(2 \cdot (\Delta(G))^k) = |kPD^*| \cdot H(2 \cdot (\Delta(G))^k).
\]
Since $H(p) \leq \ln(p) + 1$, we have
\[
|kPD| \leq |kPD^*| \cdot (1 + \ln(2 \cdot (\Delta(G))^k)) = |kPD^*| \cdot (1 + \ln 2 + k \cdot \ln(\Delta(G))). \tag{\ref{eq:approx_dist}}
\]

7. Distance $k$-paired-domination in degree bounded graphs

In this section, we show that $\text{Min Distance k-Paired-Dom Set}$ problem is APX-complete for for graphs of maximum degree 3. For the notion of APX-completeness, we refer to [3]. From now onwards, we call $\text{Min Distance k-Paired-Dom Set-B}$ the $\text{Min Distance k-Paired-Dom Set}$ problem restricted to graphs of maximum degree $B$. We first show that $\text{Min Distance k-Paired-Dom Set-4}$ problem is APX-complete by establishing an $L$-reduction from $\text{Min Distance 1-Paired-Dom Set-3}$ which is known to be APX-complete [10]. Note that $\text{Min Distance 1-Paired-Dom Set-3}$ is the usual $\text{Min Paireed-Dom Set-3}$ problem. Then we establish an $L$-reduction from $\text{Min Distance k-Paired-Dom Set-4}$ problem to $\text{Min Distance k-Paired-Dom Set-3}$ problem is APX-complete.

We first recall the notation of $L$-reduction [3,27]. Given two NP-optimization problem $\pi_1$ and $\pi_2$ and a polynomial time transformation $f$ from instances of $\pi_1$ to instances of $\pi_2$, we say that $f$ is an $L$-reduction if there are positive constants $\alpha$ and $\beta$ such that for every instance $x$ of $\pi_1$:

1. $\text{opt}_{\pi_2}(f(x)) \leq \alpha \cdot \text{opt}_{\pi_1}(x)$;
2. for every feasible solution $y$ of $f(x)$ with objective value $m_{\pi_2}(f(x), y) = c_2$, we can in polynomial time find a solution $y'$ of $x$ with $m_{\pi_1}(x, y') = c_1$ such that $|\text{opt}_{\pi_1}(x) - c_1| \leq \beta \cdot |\text{opt}_{\pi_2}(f(x)) - c_2|.$

To show the APX-completeness of a problem $\pi \in \text{APX}$, it is enough to show that there is an $L$-reduction from some APX-complete problem to $\pi$. The $\text{Min Paireed-Dom Set-3}$ problem is shown to be APX-complete [10].

Lemma 7.1. For a fixed integer $k > 0$, $\text{Min Distance k-Paired-Dom Set-4}$ problem is APX-complete.

Proof. By Theorem 6.3, it is clear that $\text{Min Distance k-Paired-Dom Set-4}$ problem is in APX. Given a graph $G = (V, E)$ and an integer $k$, an instance of $\text{Min Distance k-Paired-Dom Set-3}$ problem, we construct a graph $G'$ by attaching a path of length $k - 1$ to each vertex $v \in V$. We can easily prove that $|kPD^*| = |PD^*|$, where $kPD^*$ and $PD^*$ are optimal solutions of $G'$ and $G$, respectively. Given a distance $k$-paired-dominating set $kPD$ of $G$, we can construct a distance $k$-paired-dominating set $kPD'$ of $G'$ such that $kPD' \subseteq V(G)$. Now $kPD'$ must be a paired-dominating set of $G$. Let $PD' = kPD$. So $|PD'| \leq |kPD'|$ and hence $|PD'| - |PD^*| \leq |kPD'| - |kPD^*|$. Hence it is an $L$-reduction with $\alpha = 1$ and $\beta = 1$.

Theorem 7.2. $\text{Min Distance k-Paired-Dom Set-3}$ problem is APX-complete.

Proof. By Theorem 6.3, it is clear that $\text{Min Distance k-Paired-Dom Set-4}$ problem is in APX. Now we establish an $L$-reduction $f$ from $\text{Min Distance k-Paired-Dom Set-4}$ problem to $\text{Min Distance k-Paired-Dom Set-3}$ problem. Given a graph $G = (V, E)$ of maximum degree 4, we construct a graph $G' = (V', E')$ of maximum degree 3 as follows: for each $v$ of degree 4 in $G$, we split it and transform as shown in Fig. 3.

Let $kPD$ be a distance $k$-paired-dominating set of $G$. We can construct a $kPD'$ of $G'$ as follows:

(i) if $d_G(v) \leq 3$, then include $v$ in $kPD'$ if and only if $v \in kPD$.
(ii) if $d_G(v) = 4$ and $v \in kPD$, then include $v_{2k+2}, v_{2k+3}$ (or $v_1, v_2$) in $kPD'$.
(iii) Suppose $d_G(v) = 4$ and $v \notin kPD$. Let $u \in kPD$ such that $d_G(u, v) \leq k$ and $d_G(u, v)$ is minimum, say $k'$. Clearly $k' \in \{1, 2, \ldots, k\}$. Include $v_{2k-k'+2}, v_{2k-k'+3}$ (or $v_{2+k'}, v_{1+k'}$) in $kPD'$.
It can be easily seen that $kPD'$ is a distance $k$-paired-dominating set of $G'$ such that \(|kPD'| = |kPD| + 2 \cdot s$, where $s$ is the number of vertices of $G$ with degree 4. In particular, \(|kPD^*| \leq |kPD'| + 2 \cdot s$, where $kPD^*$ and $kPD^*$ are optimal distance $k$- paired-dominating sets of $G$ and $G'$, respectively. Since $G$ is a graph with maximum degree 4, by Lemma 2.4, we have $|kPD^*| \geq (2^{k+1} - 1) \cdot |V|$. Hence $|kPD^*| \leq |kPD'| + 2|V| \leq |kPD'| + (2^{k+1} - 1)|kPD^*| = 2^{k+1} \cdot |kPD^*|$.

Let $kPD'$ be a distance $k$-paired-dominating set of $G'$. We construct a distance $k$-paired-dominating set $kPD$ of $G$ as follows. For each vertex $v \in V(G)$ of degree 4, we include $v$ in $kPD$ if and only if $\ell(v) \geq 3$, where $\ell(v) = |kPD' \cap \{v_1, v_2, \ldots, v_{2k+3}\}|$. It is possible that the above vertices may not construct a distance $k$-paired-dominating set of $G$. Thus, finally we include a neighbor of some vertex which is already in $D$. However, in any case, $|kPD| \leq |kPD'|-2\cdot s$. In particular, $|kPD^*| \leq |kPD^*|-2\cdot s$.

Therefore, $|kPD^*| = |kPD'| + 2 \cdot s$. Now, we have $|kPD| - |kPD^*| \leq |kPD'| - 2 \cdot s - (|kPD^*| - 2 \cdot s) = |kPD'| - |kPD^*|$. Therefore, $f$ is an $L$-reduction with $\alpha = 2^{k+1}$ and $\beta = 1$. \qed

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References