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Abstract

In this paper, an attempt has been made to understand the dynamics of a prey–predator system with multiple time delays where the predator population is regarded as a generalist type. In this regard, we consider a modified Holling–Tanner prey–predator system where a constant time delay is incorporated in the logistic growth of the prey to represent a delayed density dependent feedback mechanism and the second time delay is considered to account for the length of the gestation period of the predator. Predator's interference in prey–predator relationship provides better descriptions of predator's feeding over a range of prey–predator abundances, so the predator's functional response is considered to be Type II ratio-dependent and foraging efficiency of predator largely varies with the refuge strategy of prey population. In accordance with previous studies, it is observed that delay destabilizes the system, in general and stability loss occurs via Hopf-bifurcation. In particular, we show that there exists critical values of the delay parameters below which the coexistence equilibrium is stable and above which it is unstable. Hopf bifurcation occurs when the delay parameters cross their critical values. Also, environmental stochasticity in the form of Gaussian white-noise plays a significant role to describe the system and its values. Numerical computation is also performed to validate and visualize different theoretical results presented. The analysis and results in this work are interesting both in mathematical and biological point of views.

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1. Introduction

To study the effect of environmental noise on population dynamics, researchers [1,2] have used Gaussian white-noise as a model of environmental variations. May [2] analyzed a biological system under stochastic fluctuation considering white noise for population and observed that when the population deviates more from equilibrium point, the system shows irregular behavior (i.e., instability). Ripa et al. [3] examined the effect of environmental noise on populations and presented a general theory of environmental noise in ecological food webs. Upadhyay et al. [4] investigated the influence of environmental noise on a fairly realistic ecological model with generalist top predator and shown the importance of the noise amplitude, the trophic level and the susceptibility of populations to environmental noise. Schwartz et al. [5] studied the effect of delay on the rates of noise-induced switching between co-existing stable states and noise-induced extinction in a population dynamic model.

Modeling of interacting population has begun to supplement models of evolutionary processes in addition of stochasticity at some level of interactions while sometimes the origin of stochasticity can be related to the finiteness of population size [6,7].
that defensive alliances can emerge if the chain length is more than 3. Moreover, predator–prey interactions can emerge spontaneously in evolutionary settings relevant to public goods, as reported in Correlation of positive and negative reciprocity fails to confer an evolutionary advantage [16]. They have studied a spatial public goods game, where in addition to the three strategies of defection, rewarding and punishment, a fourth strategy that combines the latter two competes for space. The impact of reward on the evolution of cooperation in the spatial public goods game is investigated and found that moderate rewards may promote cooperation better than high rewards, which is due to the spontaneous emergence of cyclic dominance between the three strategies [17]. Recently, Szolnoki et al. [18] introduced a spatial ultimatum game with discrete strategies and shown the occurrence of traveling waves and cyclic dominance where one strategy in the cycle can be an alliance of two strategies. While empathy and spatiality may lead to the evolution of fairness that would be driven by pattern formation. Finally, this subject received a recent comprehensive review on cyclic dominance in evolutionary games by Szolnoki et al. [19], focusing the pattern formation, the impact of mobility and the spontaneous emergence of cyclic dominance. Also, they highlighted the importance and usefulness of the statistical physics for the successful study of large-scale ecological systems. The potential applicability of the proposed theory extends also to the physics of social systems [20] as well as to statistical mechanics of evolutionary and coevolutionary games as reviewed in collective behavior and evolutionary games – an introduction [21].

Complex dynamical behavior arises as a consequence of time delay in a biological systems (with significant time delay) may exhibit limit cycle oscillations and chaos [22]. The larger value of gestation time delay cause individual population density to fluctuate and hence the system becomes unstable. As the estimated length of delay to preserve stability and the critical length of time delay for Hopf-bifurcation are dependent on the system’s parameter, it is possible to impose some control, which will prevent the possible abnormal oscillation in the population density. Zhao et al. [23] studied a ratio-dependent model with two time delays and obtained the estimated length of gestation delay which would not affect the stable co-existence of both the species at their equilibrium value. Dynamics of a non-delayed ratio-dependent model with constant and quadratic predator harvesting and the bifurcation of ratio-dependent system with constant rate harvesting has been reported in [24–27]. The system exhibits interesting dynamics around the coexistence equilibria, including multiple bifurcation periodic solution and homoclinic orbit. Ruan and Wei [28] studied the periodic solutions of a planar systems with two delays. Nindjin and Aziz-Alaoui [29] analyzed a prey– predator model with modified Leslie–Gower and Holling type II scheme with time delay and obtained a sufficient condition for global stability. In 2008, Yafia et al. [30] obtained the limit cycle for small and large delay in the model with Leslie Gower and Holling type II scheme. Xu and Ma [31] studied the stability and Hopf-bifurcation in a ratio-dependent system with stage-structure. Xu et al. [32] studied the stability and bifurcation of a ratio-dependent model with time delay due to gestation of the predator. Recently, Feng [33] studied the dynamics of a delayed ratio-dependent model with quadratic harvesting. Karaoglu and Merdan [34] studied a detailed Hopf-bifurcation analysis of a ratio-dependent predator-predator system involving two different discrete delays.

There has been a great deal of research on the effect of prey refuges on the population dynamics: (i) first, it affects positively the growth of prey and that of predator, comprise the reduction of prey mortality due to decrease in predation success, (ii) second the trade-off and byproducts of the hiding behavior of prey which could be detrimental for all the interacting population, and (iii) third, the refuges which protect a constant number of prey, have a stronger stabilizing effect on population dynamics than the refuges, which protect a constant proportion of prey. Magalhães et al. [35] carried out a greenhouse experiment on larvae of western flower thrips, Frankliniella occidentalis use the web produced by spider mites as refuge from predation by predatory mite, Neoseiulus cucumeris to incorporate the benefits of refuge use and develop a prey– predator model. This model predicted a minor effect of the refuge on the prey density at equilibrium. Jana et al. [36] described a time delayed prey-predator system incorporating prey refuge with Holling type II functional response. Ko and Ryu [37] investigated the asymptotic behavior of spatially inhomogeneous solution and local existence of periodic solution under the homogeneous Neumann boundary condition in a model with Holling type II functional response incorporating a prey refuge. Guan et al. [38] investigated the spatiotemporal dynamics of a 2D prey– predator model based on modified version of Leslie–Gower scheme incorporating a prey refuge. Lian et al. [39] studied the effect of time delay and cross diffusion on the spatiotemporal dynamics of a modified Leslie–Gower model incorporating a prey refuge.

In this work, we have designed a modified Holling–Tanner predator-prey model under ratio-dependent scheme with prey refuge and two time delays. The predator population is regarded as generalist type predator which switches to an alternative food option as and when it faces difficulty to find its favorite food. The positive aspects of this formulation of prey– predator interaction is that it takes care of our inability to write down growth equations for all the species on which the generalist predator feeds upon [40]. A constant time delay is incorporated in the logistic growth of prey population to represent a delayed density dependent feedback mechanism and second time delay is considered to account for the length of the gestation period of the predator. We have also analyzed the deterministic model incorporating the environmental fluctuation in the natural growth rate of prey and predator in the form of Gaussian white noise. The paper is organized in a following way: in the next section, we develop the model system and analyzed it for both delay and non-delay case. The global stability analysis of the delay model system is presented in Section 3. In Sections 4 and 5, we perform the stochastic scenario of non-delayed and delayed system respectively. Numerical results are presented in Section 6. Conclusion and discussion are presented in the last section.
2. The Mathematical model

Consider a mathematical model in which the prey population, \( x(t) \) is assumed to grow logistically to its carrying capacity \( K \) with intrinsic growth rate \( r \) in the absence of predator. The predator population, \( y(t) \) is a generalist predator and is modeled as modified version of the Leslie–Gower formulation with Holling type II functional response. Predation process follows Type II ratio-dependent functional response \([24,25,36,41]\) because a ratio-dependent prey–predator model does not show the so called paradox of enrichment \([41–43]\) and biological control paradox \([44]\). The model system is written as

\[
\begin{align*}
\frac{dx}{dt} &= r x \left( 1 - \frac{x(t - \tau_1)}{K} \right) - \frac{\alpha (1 - m) x y}{a y + (1 - m) x}, \\
\frac{dy}{dt} &= s y \left( 1 - \frac{h y(t - \tau_2)}{k + (1 - m) x(t - \tau_2)} \right).
\end{align*}
\]

(1)

The parameters appearing in the model have following meaning: \( \alpha \) is the maximal per capita prey consumption rate, \( a \) is the amount of prey necessary for the relative biomass growth rate of the predator to be half of its maximum, the predator consumes the prey according to the ratio-dependent type functional response. The parameter \( m \in (0, 1) \), is constant measures of the degree or strength of prey refuge to consider a refuge protecting \( m x \) of the prey which leaves \( (1 - m) x \) of the prey available to the predator. \( h \) is the measure of the food quality that the prey provides, which is converted to predator birth. \( k \) measures the extent to which the environment provides protection to prey population. \( s \) is the intrinsic growth rate of the generalist predator.

The initial conditions of model system (1) are given as

\[
\begin{align*}
x(\theta) &= \phi_1(\theta) \geq 0, \\
y(\theta) &= \phi_2(\theta) \geq 0, \\
\theta &\in [- \max\{\tau_1, \tau_2\}, 0], \\
\phi_i(0) &> 0 \quad (i = 1, 2),
\end{align*}
\]

(2)

where \( \phi : [- \max\{\tau_1, \tau_2\}, 0] \rightarrow \mathbb{R}^2 \) with norm

\[
||\phi|| = \sup_{-\tau \leq \theta \leq 0} \{ ||\phi_1(\theta)||, ||\phi_2(\theta)|| \}.
\]

such that \( \phi = (\phi_1, \phi_2) \).

Now, we will present some preliminaries, such as positive invariance, boundedness of solutions and the existence of equilibrium points.

2.1. Positive invariance

Feasibility or biologically positivity studies aim to objectively and rationally uncover the strengths and weaknesses of an existing proposed model in the given environment. Therefore, it is important to show positivity for the model system (1) as the system represents prey–predator population. Biologically, positivity insures that population never become negative and population always survive. Boundedness, whether a situation has a clearly defined beginning or end. The boundedness may be interpreted as a natural restriction to growth as a consequence of limited resources. For proving this, we have the following theorem.

**Theorem 2.1.** All the solution of (1) with initial conditions (2) are positive.

**Proof.** The model (1) can be written in the following form:

\[
W = \text{col}(x, y) \in \mathbb{R}_+^2, \quad (\phi_1(\theta), \phi_2(\theta)) \in C_+ = [- \max\{\tau_1, \tau_2\}, 0], \mathbb{R}_+^2.
\]

\[
\phi_1(0), \phi_2(0) > 0.
\]

\[
F(W) = \begin{pmatrix} F_1(W) \\ F_2(W) \end{pmatrix} = \begin{pmatrix} x \left( r \left( 1 - \frac{x(t - \tau_1)}{K} \right) - \frac{\alpha (1 - m) x y}{a y + (1 - m) x} \right) \\ y \left( 1 - \frac{h y(t - \tau_2)}{k + (1 - m) x(t - \tau_2)} \right) \end{pmatrix}.
\]

The model system (1) becomes

\[
\dot{W} = F(W).
\]

(3)
with \( W(\theta) = (\phi_1(\theta), \phi_2(\theta)) \in C_+ \) and \( \phi_1(0), \phi_2(0) > 0 \). It is easy to check in system (3) that whenever choosing \( W(\theta) \in \mathbb{R}_+ \) such that \( x = y = 0 \), then
\[
F(W) \big|_{W=0,W \in \mathbb{R}^2_+} \geq 0,
\]
with \( w_1(t) = x(t), w_2(t) = y(t) \). Using the lemma given in [45], any solution of (3) with \( W(\theta) \in C_+ \), say \( W(t) = W(t, W(\theta)) \), is such that \( W(t) \in \mathbb{R}^2_+ \) for all \( t \geq 0 \). Hence the solution of the system (3) exist in the region \( \mathbb{R}^2_+ \) and all solutions remain non-negative for all \( t > 0 \). Therefore, the positive orthant \( \mathbb{R}^2_+ \) is an invariant region. \( \square \)

2.2. Boundedness

Since the right hand side of system (1) is continuous and locally Lipschitzian on \( C([-\tau, 0]; \mathbb{R}^2; \tau = \max(\tau_1, \tau_2)) \), the Banach space of continuous mapping from \([-\tau, 0] \rightarrow \mathbb{R}^2 \), the solution \((x(t), y(t))\) of (1) with initial conditions \( x(\theta) > 0, \ y(\theta) > 0, \ \forall \theta \in [-\tau, 0] \) exists and is unique on \([0, \zeta]\), where \( 0 < \zeta \leq +\infty \). From the previous discussions, it is evident that \( x(t) > 0, y(t) > 0, \forall t \geq 0 \).

**Theorem 2.2.** Let \( \psi(t) = (x(t), y(t)) \), with initial conditions (2), denote any positive solution of system (1). Then there exist constants \( M_1, M_2 > 0 \) and \( T > 0 \), such that \( x(t) \leq M_1, y(t) \leq M_2 \), for \( t > T \), where \( M_1 = Ke^{\tau_1}, \ M_2 = \frac{k+11-mK}{h} e^{\tau_2} \).

**Proof.** We have, from the prey equation,
\[
\dot{x}(t) < r x(t),
\]
thus, for \( t > \tau_1 \)
\[
x(t) \leq x(t - \tau_1) e^{rt_1},
\]
which is equivalent to
\[
x(t - \tau_1) \geq x(t) e^{-rt_1}, \quad \text{for} \ t > \tau_1.
\]
Therefore, for \( t > \tau_1 \), we have
\[
\dot{x}(t) < x(t) \left( r - \frac{r}{K} e^{-rt_1} x(t) \right)
\]
A standard comparison argument shows that
\[
\limsup_{t \to +\infty} x(t) \leq Ke^{\tau_1}.
\]
Hence, if there exist \( T_1 > 0 \), such that, for \( t > T_1 \)
\[
x(t) \leq M_1.
\]
From the second equation of system (1), we have
\[
\dot{y}(t) < s y(t),
\]
thus, for \( t > \tau_2 \),
\[
y(t) \leq y(t - \tau_2) e^{s\tau_2},
\]
\[
y(t - \tau_2) \geq y(t) e^{-s\tau_2}.
\]
Again, we observe that there exits \( T_2 \) such that, for \( t \geq T_2, y(t) \leq M_2 \). Therefore, for \( t \geq T_2 + \tau_2 \), we have
\[
\dot{y}(t) < s y(t) \left( 1 - \frac{hy(t) e^{-s\tau_2}}{K + (1 - m)K} \right)
\]
Therefore, it follows that every nonnegative solution \( \psi(t) = (x(t), y(t)) \), satisfies
\[
\limsup_{t \to +\infty} y(t) \leq \frac{k + (1 - m)K}{h} e^{s\tau_2}.
\]
Hence, there exist \( M_1 > 0, M_2 > 0 \) and \( T > 0 \) such that \( x(t) \leq M_1 \) and \( y(t) \leq M_2 \), for \( t > T \). \( \square \)

2.3. Equilibria analysis and Hopf-bifurcation results

The above model (1) has five non-negative equilibria:

(i) Trivial equilibrium \( E_0(0, 0) \). The eigenvalues of the variational matrix around \( E_0 \) are \( r \) and \( s \). Therefore, \( E_0 \) is an unstable.

(ii) Predator-free axial equilibrium \( E_1(K, 0) \). The eigenvalues of the variational matrix around \( E_1 \) are \( s \) and \( -r \). Therefore, \( E_1 \) is a saddle point.
(iii) Prey-free axial equilibrium $E_2(0, \frac{k}{h})$. The eigenvalues of the variational matrix around $E_2$ are $s$ and $t - \frac{\alpha(1-m)}{a}$. Therefore, $E_2$ is also a saddle point if $ra < \alpha(1-m)$ or unstable if $ra > \alpha(1-m)$. Otherwise, $E_2$ becomes a non-hyperbolic point.

(iv) The system (1) has two coexistence equilibrium points $E_i(x_i, y_i)$ and $E^*(x^*, y^*)$, where $x_i = \frac{-Q - \sqrt{Q^2 - 4PR}}{2P}$, $y_i = \frac{k(1-m)x_i}{h}$ and $x^* = \frac{-Q + \sqrt{Q^2 - 4PR}}{2P}$, $y^* = \frac{k(1-m)x^*}{h}$.

Ecological stability can refer to types of stability in a continuum ranging from regeneration via resilience (returning quickly to a previous state), to constancy to persistence. The precise definition depends on the ecosystem in question, the variable or variables of interest, and the overall context. In the context of conservation ecology, stable population are often defined as ones that do not go extinct. Researchers applying mathematical models from system dynamics usually use Lyapunov stability. Local stability indicates that a system is stable over small short-lived disturbances. Now, we shall observe the stability features of the system locally. So, we linearizing the system (1) at $E^*(x^*, y^*)$, we get

$$\begin{align*}
\frac{dx}{dt} &= a_{11}x + a_{12}y + b_{11}x(t - \tau_1), \\
\frac{dy}{dt} &= c_{21}x(t - \tau_2) + c_{22}y(t - \tau_2),
\end{align*}$$

where,

$$a_{11} = \frac{\alpha(1-m)^2x^*y^*}{ay^* + (1-m)x^*}^2, \quad a_{12} = -\frac{\alpha(1-m)^2x^*y^*}{ay^* + (1-m)x^*}^2, \quad b_{11} = -\frac{rx^*}{K}, \quad c_{21} = \frac{sh(1-m)y^2}{k + (1-m)x^*}^2 \quad \text{and} \quad c_{22} = -\frac{shy^*}{k + (1-m)x^*}.$$

The corresponding characteristic equation is given by

$$\lambda^2 + A\lambda + C_1\lambda e^{-\lambda\tau_1} + (C_2\lambda + D_2)e^{-\lambda\tau_2} + Ee^{-\lambda(\tau_1 + \tau_2)} = 0. \quad (6)$$

where

$$A = -a_{11}, \quad C_1 = -b_{11}, \quad C_2 = -c_{22}, \quad D_2 = a_{11}c_{22} - a_{12}c_{21} \quad \text{and} \quad E = b_{11}c_{22}.$$

We now discuss the following cases.

**Case I:** $\tau_1 = 0 = \tau_2$

First, we consider when the system responses instantaneously, i.e., when the negative feedback delay in prey's logistic growth term and gestation delay of predator are both zero, in this case, the Eq. (6) becomes

$$\lambda^2 + (A + C_1 + C_2)\lambda + (D_2 + E) = 0. \quad (7)$$

All roots of the Eq. (7) will have negative real parts if and only if

$$(H_1) \quad A + C_1 + C_2 > 0 \quad \text{and} \quad D_2 + E > 0.$$

**Theorem 2.3.** The interior equilibrium point $(x^*, y^*)$ of the system (1) exists and becomes locally asymptotically stable in absence of delays if $r > \frac{a}{\alpha h}$ and $(H_1)$ hold.

**Case II:** $\tau_1 \neq 0$ and $\tau_2 = 0$

Staying at the previous situation, if prey's growth becomes delayed, i.e., if negative feedback delay of prey becomes positive, then the characteristic Eq. (6) becomes

$$\lambda^2 + (A + C_2)\lambda + D_2 + (C_1\lambda + E)e^{-\lambda\tau_1} = 0. \quad (8)$$

Let $iso(\omega > 0)$ be a root of the Eq. (8). Then we have,

$$E\cos\omega\tau_1 + C_1\omega\sin\omega\tau_1 = \omega^2 - D_2, \quad C_1\omega\cos\omega\tau_1 - E\sin\omega\tau_1 = -(A + C_2)\omega. \quad (9)$$

This leads to

$$\omega^4 - [C_1^2 - (A + C_2)^2 + 2D_2\omega^2 + D_2^2 - E^2] = 0. \quad (10)$$

It follows that the Eq. (10) has no positive roots if the following conditions are satisfied:

$$(H_2) \quad (A + C_2)^2 - C_1^2 - 2D_2 > 0 \quad \text{and} \quad D_2^2 - E^2 > 0.$$

Hence, all roots of the Eq. (10) will have negative real parts when $\tau_1 \in [0, \infty)$ if conditions of the Theorem 2.3 and $(H_2)$ are satisfied. Let

$$(H_3) \quad D_2^2 - E^2 < 0.$$
If Theorem 2.3 and (H2) hold then the Eq. (10) has a unique positive root $\tilde{\omega}_0$. Substituting $\tilde{\omega}_0$ into Eq. (9), we have

$$\tilde{\tau}_1 = \frac{1}{\tilde{\omega}_0} \cos^{-1}\left[ \frac{E(\tilde{\omega}_0^2 - D_2) - (A + C_2)C_1 \tilde{\omega}_0^2}{C_1^2 \tilde{\omega}_0^2 + E^2} \right] + \frac{2p\pi}{\tilde{\omega}_0}, \quad p = 0, 1, 2, \ldots,$$

where $\tilde{\omega}_0$ is the unique positive root of Eq. (10). Let,

\[(H_4) \quad (A + C_2)^2 - C_1^2 - 2D_2 < 0, \quad D_2^2 - E^2 > 0.\]

If (H1) and (H4) hold, then Eq. (10) has two positive roots $\tilde{\omega}_0^+ \text{ and } \tilde{\omega}_0^-$. Substituting $\tilde{\omega}_0^\pm$ into Eq. (9) we obtain

$$\tilde{\tau}_{1\pm} = \frac{1}{\tilde{\omega}_0^\pm} \cos^{-1}\left[ \frac{E(\tilde{\omega}_0^\pm^2 - D_2) - (A + C_2)C_1 \tilde{\omega}_0^\pm^2}{C_1^2 \tilde{\omega}_0^\pm^2 + E^2} \right] + \frac{2q\pi}{\tilde{\omega}_0^\pm}, \quad q = 0, 1, 2, \ldots.$$

If $\lambda(\tau_1)$ is a root of Eq. (8) satisfying $\text{Re}(\lambda(\tau_1)) = 0$ (or $\text{Re}(\lambda(\tau_1^\pm)) = 0$, respectively) and $\text{Im}(\lambda(\tau_1)) = 0$ (or $\text{Im}(\lambda(\tau_1^\pm)) = \tilde{\omega}_0^\pm$, respectively), we obtain

$$\left[ \frac{d}{d\tau_1}(\text{Re}(\lambda(\tau_1))) \right]_{\tilde{\tau}_{1\pm}, \tilde{\omega} = \tilde{\omega}_0} > 0, \quad \left[ \frac{d}{d\tau_1}(\text{Re}(\lambda(\tau_1))) \right]_{\tilde{\tau}_{1\pm}, \tilde{\omega} = \tilde{\omega}_0} < 0.$$

**Theorem 2.4.** Assume $\tau_1 \neq 0$, $\tau_2 = 0$ and conditions of the Theorem 2.1 are satisfied, then the equilibrium $E^*(x^*, y^*)$ is locally asymptotically stable for $\tau_1 < \tilde{\tau}_{10}$ and unstable for $\tau_1 > \tilde{\tau}_{10}$. Furthermore, the system (1) undergoes a Hopf-bifurcation at $(x^*, y^*)$ when $\tau_1 = \tilde{\tau}_{10}$, where

$$\tilde{\tau}_{10} = \frac{1}{\tilde{\omega}_0} \cos^{-1}\left[ \frac{E(\tilde{\omega}_0^2 - D_2) - (A + C_2)C_1 \tilde{\omega}_0^2}{C_1^2 \tilde{\omega}_0^2 + E^2} \right].$$

**Case III:** $\tau_1 \in (0, \tilde{\tau}_{10}), \tau_2 \neq 0$.

In this case, we allow a gestation time period for the predator and also a constant time delay affecting the density dependent growth rate of the prey. We fix $\tau_1$ at some value from its stability range $(0, \tilde{\tau}_{10})$ and regard $\tau_2$ as a free parameter. We also assume that the model parameters are such that hypothesis (H1) holds. Let $i\omega$ with $\omega > 0$ be a root of Eq. (6). Then

$$\omega^4 + \tilde{a}\omega^2 + 2\tilde{b}\sin(\omega\tau_1) + 2\tilde{c}\cos(\omega\tau_1) + \tilde{d} = 0,$$

where

$$\tilde{a} = A^2 + C_1^2 - C_2^2, \quad \tilde{b} = \omega C_2 E - \omega^3 C_1,$$

$$\tilde{c} = \omega D_2 E - \omega^2 D_1 + \omega^2 A C_1, \quad \tilde{d} = -(D_2^2 + E^2).$$

Note that $\tilde{d}$ is always negative. We define

$$f(\omega) = \omega^4 + \tilde{a}\omega^2 + 2\tilde{b}\sin(\omega\tau_1) + 2\tilde{c}\cos(\omega\tau_1) + \tilde{d}.$$

Then it is easy to verify that $f(0) < 0$ and $f(\omega) \to \infty$, as $\omega \to \infty$. Thus, Eq. (11) has finitely many positive roots, denoted by: $\tilde{\omega}_1, \tilde{\omega}_2, \ldots, \tilde{\omega}_k$. For every fixed $\tilde{\omega}_s$, where $s = 1, 2, \ldots, k$, there exists a sequence $\{\tilde{\omega}_{2t} | t \in \mathbb{N}\}$, where

$$\tilde{\tau}_{2t} = \frac{1}{\tilde{\omega}_s} \cos^{-1}\left( \frac{M_2}{N_2} \right) + \frac{2s\pi}{\tilde{\omega}_s}, \quad s = 1, 2, \ldots, k; \quad t \in \mathbb{N},$$

with

$$M_2 = P_1 S_2 + P_2 T_2 + R_2 T_2 \cos \tilde{\omega}_s \tau_1 + R_2 S_2 \sin \tilde{\omega}_s \tau_1,$$

$$N_2 = S_2^2 + T_2^2, \quad R_1 = -\tilde{\omega}_s^2, \quad P_2 = A \tilde{\omega}_s, \quad R_2 = C_1 \tilde{\omega}_s,$$

$$S_2 = -(E \cos \tilde{\omega}_s \tau_1 + D_2), \quad T_2 = E \sin \tilde{\omega}_s \tau_1 - C_2 \tilde{\omega}_s, \quad s = 1, 2, \ldots, k;$$

such that Eq. (11) holds. Let $\tilde{\tau}_{20} = \min\{\tilde{\tau}_{2t} | s = 1, 2, \ldots, k; \quad t = 1, 2, \ldots\}$. When $\tau_2 = \tilde{\tau}_{20}$, Eq. (6) has a pair of pure imaginary roots $\pm i\tilde{\omega}_1$ for $\tau_2 \in (0, \tilde{\tau}_{20})$. In the following, we assume that

$$(H_5) \quad \left[ \frac{d}{d\tau_2}(\text{Re}(\lambda(\tau_2))) \right]_{\lambda = i\tilde{\omega}_1} \neq 0.$$
Therefore, by the general Hopf-bifurcation theorem of functional differential equations [46], we obtain the following result considering the change of stability of the model system (1).

**Theorem 2.5.** Suppose that the parameters in model system (1) are such that hypotheses \((H_1)\) and \((H_2)\) hold true, and \(\tau_1 \in [0, \bar{\tau}_1]\). Then the coexistence equilibrium \(E^*(x^*, y^*)\) is locally asymptotically stable when \(\tau_2 < \bar{\tau}_2\) and unstable for \(\tau_2 > \bar{\tau}_2\). Moreover, Hopf bifurcation occurs when \(\tau_2 = \bar{\tau}_2\).

**Case IV:** \(\tau_1 = 0\) and \(\tau_2 \neq 0\). For this choice of the delay parameters we summarize our results in the following theorem. The proof follows similar arguments as the stability theorems **Theorem 2.4.** in Case II.

**Theorem 2.6.** Assume that \(\tau_1 = 0\), \(\tau_2 \neq 0\) and the conditions in hypothesis \((H_1)\) hold true. Then the equilibrium \(E^*(x^*, y^*)\) is locally asymptotically stable for \(\tau_2 < \bar{\tau}_2\) and unstable for \(\tau_2 > \bar{\tau}_2\). Furthermore, the system (1) undergoes Hopf-bifurcation when \(\tau_2 = \bar{\tau}_2\), where

\[
\tau_{20} = \frac{1}{\omega_0} \cos^{-1} \left( \frac{(D_2 + E)\omega_0^2 - (A + C_1)C_2\omega_0^2}{C_2^2\omega_0^2 + (D_2 + E)^2} \right)
\]

and \(\omega_0\) is the unique positive root of the polynomial:

\[
\omega^4 - [C_2^2 - (A + C_1)^2] \omega^2 - (D_2 + E)^2 = 0.
\]

**Case V:** \(\tau_1 \neq 0\), \(\tau_2 \neq 0\) and \(\tau_2\) is within its stability range. This choice of parameter regime corresponds to the biologically interesting case, when the predator individuals have a fixed gestation period and the growth of the prey population is affected by delayed density dependent mechanism. The proof of the main result follows similar lines to that in Case III and we only summarize the stability results in the following theorem.

**Theorem 2.7.** Assume that the model parameters are such that the hypotheses in \((H_1)\) hold true and \(\tau_2 \in [0, \bar{\tau}_2]\). Then the coexistence equilibrium \(E^*(x^*, y^*)\) is asymptotically stable when \(\tau_1 \in [0, \bar{\tau}_1]\) and it is unstable when \(\tau_1 > \bar{\tau}_1\). Hopf-bifurcation occurs when \(\tau_1 = \bar{\tau}_1\), where \(\bar{\tau}_1 = \min\{\tau_1^j, k \in \mathbb{N}\}\) and

\[
\tau_1^j = \left( \frac{1}{\omega_1} \right) \cos^{-1} \left( \frac{M_1}{N_1} \right) + \frac{2\pi}{\omega_1}, \quad i = 1, 2, \ldots, k; \quad j \in \mathbb{N};
\]

with

\[
M_1 = P_1S_1 + P_2T_1 + (Q_1S_1 + R_1T_1) \cos(\omega_1\tau_2) + (R_1S_1 - Q_1T_1) \sin(\omega_1\tau_2),
\]

\[
N_1 = S_1^2 + T_1^2, \quad P_1 = -\omega_1^2, \quad P_2 = \omega_2^2, \quad Q_1 = D_2, \quad R_1 = C_2\omega_1, \quad S_1 = -E \cos(\omega_1\tau_2), \quad T_1 = E \sin(\omega_1\tau_2) - C_1\omega_1, \quad i = 1, 2, \ldots, k.
\]

3. **Global stability analysis**

Observational studies of ecosystems use constancy to describe living systems that can remain unchanged. Global stability indicates a system highly resistant to change in species composition and/or food web dynamics. Here \(x, y \in \mathbb{R}^2\) are the state variables with inputs as written in Eq. (2) which can be usually partitioned into manipulable controls and exogenous disturbances, according to different set-ups. Global asymptotic stability of an invariant compact set \(\Omega\) is defined as local stability along with global attractivity.

**Theorem 3.1.** If \(\min\{p_1, p_2\} > 0\), with

\[
p_1 = \frac{r}{K} - \frac{r^2M_1T_1}{K^2} - \frac{\alpha(1 - m)}{aM_1} - \frac{rM_1\alpha(1 - m)}{K\alpha M_1} - \frac{sh^r}{k} \left( 1 + \frac{sh^r T_2}{k} \right),
\]

\[
p_2 = sh^r \left( \frac{1}{k + (1 - m)M_1} - \frac{sh^r T_2}{k^2} \right) - \frac{\alpha}{m_1} \left( 1 + \frac{r\tau_1 M_1}{K} \right),
\]

where \(m_1 < x(t) \leq M_1\) and \(m_2 < y(t) \leq M_2\) for \(t > 0\), then the interior equilibrium point of the system (1), is globally asymptotically stable.

**Proof.** The method used to proof this theorem is to construct a suitable Lyapunov function to derive sufficient condition which guarantee that the positive interior equilibrium \(E^*(x^*, y^*)\) of the system (1) is globally asymptotically stable. For mathematical convenience, we make the following transformations of the variables:

\[
x(t) = x^* e^{rt}, \quad y(t) = y^* e^{rt}.
\]
These coordinate change transforms the positive equilibrium $E_2$ into the trivial equilibrium $u(t) = v(t) = 0$ for all $t > 0$. Due to the above transformations, the system (1) is reduced as follows:

\[
\frac{du}{dt} = -\frac{rx^*}{K} (e^{u(t)} - 1) - \frac{\alpha (1 - m)^2 x^* y^* (e^{u(t)} - 1)}{(ay^* + (1 - m)x^*)(ay + (1 - m)x)} + \frac{\alpha (1 - m)^2 x^* y^* (e^{u(t)} - 1)}{(ay^* + (1 - m)x^*)(ay + (1 - m)x)}. \tag{13}
\]

\[
\frac{dv}{dt} = \frac{sh(1 - m)x^* y^*}{(k + (1 - m)x^*)(k + (1 - m)x(t - \tau))} (e^{v(t)} - 1) - \frac{shky^*}{(k + (1 - m)x^*)(k + (1 - m)x(t - \tau))} \times (e^{v(t)} - 1).
\tag{14}
\]

By using the following relation

\[
e^{u(t)} = e^{u(t)} - \int_{t-\tau}^t e^{u(s)} \frac{du}{ds} ds.
\]

Eq. (13) can be rewritten as

\[
\frac{du}{dt} = -\frac{rx^*}{K} (e^{u(t)} - 1) - \frac{\alpha (1 - m)^2 x^* y^* (e^{u(t)} - 1)}{(ay^* + (1 - m)x^*)(ay + (1 - m)x)} + \frac{\alpha (1 - m)^2 x^* y^* (e^{u(t)} - 1)}{(ay^* + (1 - m)x^*)(ay + (1 - m)x)}
\]

\[
+ \frac{rx^*}{K} \int_{t-\tau}^t e^{u(s)} \left[ -\frac{rx^*}{K} (e^{u(s)} - 1) - \frac{\alpha (1 - m)^2 x^* y^* (e^{u(s)} - 1)}{(ay^* + (1 - m)x^*)(ay + (1 - m)x)}
\]

\[
+ \frac{\alpha (1 - m)^2 x^* y^* (e^{u(s)} - 1)}{(ay^* + (1 - m)x^*)(ay + (1 - m)x)} \right] ds.
\tag{15}
\]

Let

\[
V_1(t) = |u(t)|.
\]

Computing the upper right derivative of $V_1(t)$ along the solution of the model system (1), it follows from Eq. (15) that

\[
D^+ V_1(t) \leq -\frac{rx^*}{K} |e^{u(t)} - 1| - \frac{\alpha (1 - m)^2 x^* y^* |e^{u(t)} - 1|}{(ay^* + (1 - m)x^*)(ay + (1 - m)x)} + \frac{\alpha (1 - m)^2 x^* y^* |e^{u(t)} - 1|}{(ay^* + (1 - m)x^*)(ay + (1 - m)x)}
\]

\[
+ \frac{rx^*}{K} \int_{t-\tau}^t \left[ -\frac{rx^*}{K} |e^{u(s)} - 1| - \frac{\alpha (1 - m)^2 x^* y^* |e^{u(s)} - 1|}{(ay^* + (1 - m)x^*)(ay + (1 - m)x)}
\]

\[
+ \frac{\alpha (1 - m)^2 x^* y^* |e^{u(s)} - 1|}{(ay^* + (1 - m)x^*)(ay + (1 - m)x)} \right] ds.
\tag{16}
\]

Again due to structure of Eq. (16), we consider the following functional

\[
V_{12}(t) = V_1(t) + \frac{rM_1}{K} \int_{t-\tau}^t \int_{s-\tau}^s \left[ \frac{rx^*}{K} |e^{u(s)} - 1| + \frac{\alpha (1 - m)^2 x^* y^* |e^{u(s)} - 1|}{(ay^* + (1 - m)x^*)(ay + (1 - m)x)}
\]

\[
+ \frac{\alpha (1 - m)^2 x^* y^* |e^{u(s)} - 1|}{(ay^* + (1 - m)x^*)(ay + (1 - m)x)} \right] ds dy + \frac{r^2 x^* \tau M_1}{K^2} \int_{t-\tau}^t |e^{u(s)} - 1| ds,
\]

whose upper right derivative along the solution of the system (1) is given by

\[
D^+ V_{12}(t) = D^+ V_1 + \frac{rM_1 \tau_1}{K} \left\{ \frac{rx^*}{K} |e^{u(t)} - 1| + \frac{\alpha (1 - m)^2 x^* y^* |e^{u(t)} - 1|}{(ay^* + (1 - m)x^*)(ay + (1 - m)x)}
\]

\[
+ \frac{\alpha (1 - m)^2 x^* y^* |e^{u(t)} - 1|}{(ay^* + (1 - m)x^*)(ay + (1 - m)x)} \right\} - \frac{rM_1}{K} \int_{t-\tau}^t \left[ \frac{rx^*}{K} e^{u(s)} - 1 \right] ds.
\]
\[
\begin{align*}
\frac{d}{dt} \mathcal{L} + \frac{r^2 \tau_1 M_1}{K^2} & \left( |e^{u(t)}| - 1 \right) - |e^{u(t-\tau_1)} - 1|, \\
= & - \frac{r^2 \tau_1 M_1}{K^2} \left( |e^{u(t)}| - 1 \right) - \frac{r M_1 \tau_1 (1-m)^2 y^*(e^{u(t)} - 1)}{K(a^* + (1-m)k^*)} \left( |e^{u(t)}| - 1 \right) \\
& - \frac{r M_1 \tau_1 (1-m)^2 y^*}{K(a^* + (1-m)k^*)} \left( 1 + \frac{r M_1 \tau_1}{K} \right) y^* |e^{u(t)} - 1|, \\
& \leq - \frac{r^2 \tau_1 M_1}{K^2} - \frac{r M_1 \tau_1 (1-m)^2 y^*}{a M_1} \left( |e^{u(t)}| - 1 \right) + \frac{r M_1 \tau_1 (1-m)^2 y^*}{a M_1} \left( 1 + \frac{r M_1 \tau_1}{K} \right) |e^{u(t)} - 1|. 
\end{align*}
\]

The second Eqs. (14) of system (1) can be rewritten as

\[
\begin{align*}
\frac{dv}{dt} &= \frac{sh(1-m)k^* y^*(e^{u(t-\tau_2)} - 1)}{(k^* + (1-m)k^*)} - \frac{shy^*(e^{u(t)} - 1)}{(k^* + (1-m)k^*)} \\
& + \frac{shy^*}{(k^* + (1-m)k^*)} \int_{t-\tau_2}^{t} e^{u(s)} \left( \frac{sh(1-m)k^* y^*(e^{u(s-\tau_2)} - 1)}{(k^* + (1-m)k^*)} - \frac{shy^*(e^{u(s)} - 1)}{(k^* + (1-m)k^*)} \right) ds. 
\end{align*}
\]

Let \( V_2 = |v(t)| \). Computing the upper right derivative of \( V_2(t) \) along the solution of the model system (1), we get

\[
\begin{align*}
D^+ V_2(t) & \leq \frac{sh(1-m)k^* y^* |e^{u(t-\tau_2)} - 1|}{(k^* + (1-m)k^*)} - \frac{shy^* |e^{u(t)} - 1|}{(k^* + (1-m)k^*)} \\
& + \frac{shy^*}{(k^* + (1-m)k^*)} \int_{t-\tau_2}^{t} e^{u(s)} \left( \frac{sh(1-m)k^* y^* |e^{u(s-\tau_2)} - 1|}{(k^* + (1-m)k^*)} - \frac{shy^* |e^{u(s)} - 1|}{(k^* + (1-m)k^*)} \right) ds. 
\end{align*}
\]

By Theorem 2.2, we see that there exists a \( \tau > 0 \), such that \( X^* e^{u(t)} \leq M_1, Y^* e^{u(t)} \leq M_2 \) for all \( t > \tau \) and for \( t \geq \tau + \tau \), we have

\[
\begin{align*}
D^+ V_2(t) & \leq \frac{shy^* |e^{u(t)} - 1|}{k} + \frac{shy^* |e^{u(t)} - 1|}{k} + \frac{shy^*}{k} \int_{t-\tau_2}^{t} \left( \frac{shy^* |e^{u(s-\tau_2)} - 1|}{k} + \frac{shy^*}{k} |e^{u(s-\tau_2)} - 1| \right) ds. 
\end{align*}
\]
Again due to structure of Eq. (21) we consider the following functional

\[
V_{22}(t) = V_2(t) + \frac{shM_2}{k} \int_{t-\tau_2}^t \left\{ \frac{shy^* e^{\theta_2(t-\tau_2)}}{k} + \frac{shy^* e^{\theta_1(t-\tau_2)}}{k} \right\} ds dy
\]

whose upper right derivative along the solution of the system (1) is given by

\[
D^+ V_{22}(t) \leq D^+ V_2 + \frac{shM_2 \tau_2}{k} \left\{ \frac{shy^* e^{\theta_2(t-\tau_2)}}{k} + \frac{shy^* e^{\theta_1(t-\tau_2)}}{k} \right\} ds
\]

\[
+ \frac{shy^*}{k} \left[ |e^{\theta_1(t)} - 1| - |e^{\theta_1(t-\tau_2)} - 1| + \frac{shy^*}{k} |e^{\theta_1(t)} - 1| - |e^{\theta_1(t-\tau_2)} - 1| \right]
\]

\[
= \frac{shy^*}{k} \left( 1 + \frac{shy^*}{k} \right) |e^{\theta_1(t-\tau_2)} - 1| - \frac{shy^*}{k} \left( \frac{1}{k+1-m} \right) |e^{\theta_1(t)} - 1|.
\]

Let us define a Lyapunov functional \( V(t) \) as

\[
V(t) = V_{12}(t) + V_{22}(t) > |u(t)| + |v(t)|.
\]

Computing the upper right derivative of \( V(t) \) along the solution of the system (1), and by using Eqs. (17) and (22), we obtain

\[
D^+ V(t) = D^+ V_{12}(t) + D^+ V_{22}(t)
\]

\[
\leq -x^* \left[ \frac{r}{K} - \frac{r^2 M_1 \tau_1}{k^2} - \frac{r M_1 \tau_1 (1-m)}{a M_1} - \frac{r M_1 \tau_1 (1-m) K M_1}{K a M_1} \right] |e^{\theta_1(t)} - 1| + \frac{r \tau_1 M_1}{K} |e^{\theta_1(t)} - 1|
\]

\[
+ \frac{shy^*}{k} \left( 1 + \frac{shy^*}{k} \right) |e^{\theta_1(t-\tau_2)} - 1| - \frac{shy^*}{k} \left( \frac{1}{k+1-m} \right) |e^{\theta_1(t)} - 1|.
\]

\[
\leq -p_1 x^* |e^{\theta_1(t)} - 1| - p_2 y^* |e^{\theta_1(t)} - 1|.
\]

Since the model system (1) is positive invariant, therefore, for all \( t > T^* \), we have

\[
x^* e^{\theta_1(t)} = x(t) > m_1, \quad y^* e^{\theta_1(t)} = y(t) > m_2.
\]

Using the mean value theorem, we have

\[
x^* |e^{\theta_1(t)} - 1| = x^* e^{\theta_1(t)} |u(t)| > m_1 |u(t)|,
\]

\[
y^* |e^{\theta_1(t)} - 1| = y^* e^{\theta_1(t)} |v(t)| > m_2 |v(t)|,
\]

where \( x^* e^{\theta_1(t)} \) lies between \( x^* \) and \( x(t) \), \( y^* e^{\theta_1(t)} \) lies between \( y^* \) and \( y(t) \). Therefore,

\[
D^+ V(t) \leq -p_1 m_1 |u(t)| - p_2 m_2 |v(t)| \leq -\eta (|u(t)| + |v(t)|), \quad \text{where} \quad \eta = \min\{p_1 m_1, p_2 m_2\}.
\]

Noting that \( V(t) > |u(t)| + |v(t)| \). Hence, by applying Lyapunov direct method for global stability theorem and Eq. (24), we can conclude that, the zero solution of the reduced system (13)–(14) is globally asymptotically stable. Therefore, the positive equilibrium of the original model system (1) is globally asymptotically stable. \( \square \)
4. Stochastic scenario of non-delayed system

Deterministic models in ecology do not usually incorporate environmental fluctuation; they are often justified by the implicit assumption that in large population, stochastic deviations are small enough to be ignored. Deterministic model will prove ecologically useful only if the dynamical patterns they reveal are still in evidence when stochastic effects are introduced. For terrestrial system, the environmental variability is large at both short and long time periods and could be expected to develop internal mechanisms to the system which would cope with short term variability and minimize the effects of long term variations, hence analysis of the system with white noise gives better results. Uncertain growth of population is usually considered as an effect of environmental stochasticity. Reproduction of species depends on various factors, such as temperature, humidity, parasites and pathogens, environmental pollution etc. [47]. Since physical and biological environments of populations are not totally predictable, the growth of population should be considered as a stochastic process rather than a deterministic one [48]. In spite of some shortcomings, Gaussian white noise has been proved extremely useful to model rapidly fluctuating phenomena [49,50]. Therefore, in the non-delayed counterpart of model system (1) by introducing the environmental stochasticity in the form of Gaussian white noise is represented by:

\[
\begin{align*}
\frac{dx}{dt} & = rx \left( 1 - \frac{x}{K} \right) - \frac{\alpha (1 - m) xy}{ay + (1 - m)x} + \alpha_1 \eta_1(t), \\
\frac{dy}{dt} & = sy \left( 1 - \frac{hy}{k + (1 - m)x} \right) + \alpha_2 \eta_2(t), \\
\end{align*}
\]  

where the perturbed terms \( \eta_1(t) \) and \( \eta_2(t) \) are assumed to be the independent Gaussian white noise satisfying the conditions:

\[
\langle \eta_j(t) \rangle = 0 \quad \text{and} \quad \langle \eta_j(t_1) \eta_j(t_2) \rangle = \delta(t_1 - t_2) \quad \text{for} \quad j = 1, 2.
\]

Here \( \delta \) are the intensities or strengths of the random perturbations (Kronecker symbol), \( \delta \) is the Dirac delta function defined by

\[
\delta(x) = 0, \quad \text{for} \quad x \neq 0,
\]

\[
\int_{-\infty}^{\infty} \delta(x) dx = 1,
\]

\( \langle \cdot \rangle \) represents the ensemble average of the stochastic process and \( \alpha_j \) are real constants.

Though Gaussian white noises are so very irregular, these are extremely useful to model rapidly fluctuating phenomena. Of course true white noises do not occur in nature. However, as can be seen by studying their spectra, thermal noises in electrical resistance, the force acting on a Brownian particle, climate fluctuations, disregarding the periodicity of astronomical origin etc. are white to a very good approximation. These examples support the usefulness of the white noise idealization in applications to natural systems. Furthermore, it can be proved that the process \((x, y)^T\), solution of Eq. (25), is Markovian if and only if the external noises are white. These results explain the importance and appeal of the white-noise idealization. It is noted that \( \eta_j(t) \) are not defined in the ordinary sense. It can be proved that \( \eta_j(t) \) are the derivatives of the Wiener process \( W(t) \) in the generalized functions sense. Assume that fluctuations in the environment will manifest themselves mainly as fluctuations in the natural growth rate of the prey and in the natural mortality rate of the predator since these are the main terms subject to coupling of a prey–predator pair with its environment.

To study the behavior of the system (25) about the steady state \( E^* \), let us substitute \( x' = \ln x, y' = \ln y; x = u + x^* \) and \( y = v + y^* \). Then the system (25) reduces to the following linear stochastic differential equations in terms of deviation variables \((u, v)\):

\[
\begin{align*}
\frac{du}{dt} & = \alpha_{11} x^* u + \alpha_{12} x^* v + \alpha_1 \eta_1(t), \\
\frac{dv}{dt} & = \alpha_{21} y^* u + \alpha_{22} y^* v + \alpha_2 \eta_2(t), \\
\end{align*}
\]  

where

\[
\begin{align*}
\alpha_{11} & = - \frac{r}{K} + \frac{\alpha (1 - m) y^*}{ay^* + (1 - m)x^*}, \\
\alpha_{12} & = - \frac{\alpha (1 - m) x^*}{ay^* + (1 - m)x^*}, \\
\alpha_{21} & = \frac{sh (1 - m) y^*}{k + (1 - m)x^*}, \\
\alpha_{22} & = - \frac{sh}{k + (1 - m)x^*}.
\end{align*}
\]  

The solutions \((u(t), v(t))\) subject to known initial values \((u(t_0), v(t_0))\) determine the statistical behavior of the model system (25) near the steady state \( E^* \) at time \( t > t_0 \).

Taking the Fourier transform on both sides of system (26), we get,

\[
\begin{align*}
\alpha_1 \tilde{\eta}_1(\omega) & = (i\omega + \alpha_{11} x^*) \hat{U}(\omega) + \alpha_{12} x^* \hat{V}(\omega), \\
\alpha_2 \tilde{\eta}_2(\omega) & = \alpha_{21} y^* \hat{U}(\omega) + (i\omega + \alpha_{22} y^*) \hat{V}(\omega).
\end{align*}
\]

The matrix of Eq. (28) is

\[
M(\omega) \hat{U}(\omega) = \tilde{\eta}(\omega)
\]
where
\[
M(\omega) = \begin{pmatrix}
i\omega + \alpha_{11}x' & \alpha_{12}x' \\
i\omega + \alpha_{21}y' & \alpha_{22}y'
\end{pmatrix}, \quad \tilde{U}(\omega) = \begin{pmatrix}\tilde{U}(\omega) \\
\tilde{v}(\omega)\end{pmatrix}, \quad \tilde{\eta}(\omega) = \begin{pmatrix}\tilde{\eta}_1(\omega) \\
\tilde{\eta}_2(\omega)\end{pmatrix}.
\]

Eq. (29) can also be written as \(\tilde{U}(\omega) = [M(\omega)]^{-1}\tilde{\eta}(\omega)\). Let \([M(\omega)]^{-1} = K(\omega)\), therefore
\[
\tilde{U}(\omega) = K(\omega)\tilde{\eta}(\omega).
\]

where
\[
K(\omega) = \frac{\text{Adj}(M(\omega))}{|M(\omega)|}.
\]

If the function \(Y(t)\) has a zero mean value, then the fluctuation intensity (variance) of it is components in the frequency interval \([\omega, \omega + d\omega]\) is \(S_Y(\omega)d\omega\), where \(S_Y(\omega)\) is spectral density of \(Y\) and is defined as
\[
S_Y(\omega) = \lim_{T \to \infty} \frac{1}{T} \left| \hat{Y}(\omega) \right|^2.
\]

The power spectrum of a time series describes how the variance of the data is distributed over the frequency components into which the data may be decomposed. Intuitively, the spectrum decomposes the content of a signal or of a stochastic process into the different frequencies present in that process and helps to identify periodicities. If \(Y\) has a zero mean value, the inverse transform of \(S_Y(\omega)\) is the auto covariance function
\[
C_Y(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(\omega)e^{i\omega\tau}d\omega.
\]

The corresponding variance of fluctuations in \(Y(t)\) is given by
\[
\sigma_Y^2 = C_Y(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Y(\omega)d\omega
\]
and the auto correlation function is the normalized auto covariance
\[
R_Y(\tau) = \frac{C_Y(\tau)}{C_Y(0)}.
\]

For a Gaussian white noise process, it is
\[
S_{\tilde{\eta}\tilde{\eta}}(\omega) = \lim_{T \to \infty} \frac{1}{T} \langle \tilde{\eta}(\omega)\tilde{\eta}(\omega) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \tilde{\eta}(t)\tilde{\eta}(t')e^{-i\omega(t-t')}dtdt' = \delta_{ij}.
\]

From Eq. (30), we have
\[
\tilde{U}(\omega) = \sum_{j=1}^{2} K_{ij}(\omega)\tilde{\eta}(\omega), \quad i = 1, 2.
\]

And from Eq. (32), we have
\[
S_U(\omega) = \sum_{j=1}^{2} \xi_j |K_{ij}(\omega)|^2, \quad i = 1, 2.
\]

Intensities of fluctuations or variance measures how far a set of numbers is spread out. A variance of zero indicates that all the values are identical. A small variance indicates that the data points tend to be very close to the mean (expected value, here the equilibrium density of prey and predator population) and hence to each other, while a high variance indicates that the data points are very spread out around the mean and from each other. Hence by Eqs. (34) and (38), the intensities of fluctuations in the variable \(U\) are given by
\[
\sigma_U^2 = \frac{1}{2\pi} \sum_{j=1}^{2} \alpha_j |K_{ij}(\omega)|^2 d\omega, \quad i = 1, 2.
\]

and by Eq. (31), we obtain
\[
\sigma_U^2 = \frac{1}{2\pi} \left[ \int_{-\infty}^{\infty} \alpha_1 \frac{A_1}{|M(\omega)|} |^2 d\omega + \int_{-\infty}^{\infty} \alpha_2 \frac{B_1}{|M(\omega)|} |^2 d\omega \right],
\]
\[
\sigma_U^2 = \frac{1}{2\pi} \left[ \int_{-\infty}^{\infty} \alpha_1 \frac{A_2}{|M(\omega)|} |^2 d\omega + \int_{-\infty}^{\infty} \alpha_2 \frac{B_2}{|M(\omega)|} |^2 d\omega \right],
\]

where \(|M(\omega)| = |R(\omega)| + i|l(\omega)|\) and real part of \(|M(\omega)|\) is
\[
R^2(\omega) = -\omega^2 + \alpha_{11}\alpha_{22}x'y' - \alpha_{12}\alpha_{21}x'y'.
\]
Imaginary part of $|M(\omega)|$ is
\[ I^2(\omega) = \alpha_{22} \omega y^* + \alpha_{11} \omega x^*, \] (42)
and
\[ |A_1|^2 = X_1^2 + Y_1^2, \quad |B_1|^2 = X_2^2 + Y_2^2, \quad |A_2|^2 = X_3^2 + Y_3^2, \quad |B_2|^2 = X_4^2 + Y_4^2, \]
where,
\[ X_1^2 = \alpha_{22} y^*, \quad Y_1^2 = \omega, \quad X_2^2 = \alpha_{21} y^*, \quad Y_2^2 = 0, \quad X_3^2 = \alpha_{12} x^*, \quad Y_3^2 = 0, \quad X_4^2 = \alpha_{11} x^* \text{ and } Y_4^2 = \omega. \]
Thus Eq. (40) becomes
\[
\sigma_u^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( I^2(\omega) + I^2(\omega) \right) d\omega,
\]
\[
\sigma_v^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \alpha_1 (X_1^2 + Y_1^2) + \alpha_2 (X_2^2 + Y_2^2) \right) d\omega.
\] (43)

If we are interested in the dynamics of system (25) with either $\alpha_1 = 0$ or $\alpha_2 = 0$, then the population variances are:
If $\alpha_1 = 0$, then
\[
\sigma_u^2 = \frac{\alpha_2 (X_2^2 + Y_2^2)}{2\pi} \int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} d\omega,
\]
\[
\sigma_v^2 = \frac{\alpha_2 (X_4^2 + Y_4^2)}{2\pi} \int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} d\omega. \] (44)

If $\alpha_2 = 0$, then
\[
\sigma_u^2 = \frac{\alpha_1 (X_1^2 + Y_1^2)}{2\pi} \int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} d\omega,
\]
\[
\sigma_v^2 = \frac{\alpha_1 (X_3^2 + Y_3^2)}{2\pi} \int_{-\infty}^{\infty} \frac{1}{R^2(\omega) + I^2(\omega)} d\omega. \] (45)

Finally, we conclude that Fourier transform method which has been used to study the effects of stochasticity on the positive equilibrium of our model system leading to chaos in a realistic ecological situation.

5. Stochastic scenario of delayed system

The prey–predator system (1) with delay in a random environment will be considered as:
\[
\frac{dx}{dt} = x \left[ r \left( 1 - \frac{x(t - \tau_1)}{K} \right) - \frac{\alpha (1 - m) x y}{ay + (1 - m)x} + \alpha_1 \eta_1(t) \right],
\]
\[
\frac{dy}{dt} = y \left[ s \left( 1 - \frac{hy(t - \tau_2)}{k + (1 - m)x(t - \tau_2)} \right) \right] + \alpha_2 \eta_2(t). \] (46)

Again using the transformations: $x' = \ln x$, $y' = \ln y$; $x = u + x'$, $y = v + y'$ and assuming the delay to be very small, the system (46) (to a first approximation) can be written as
\[
\frac{du}{dt} = \tilde{A}_1 x' u + c_1 x' v + A_1 x' u(t - \tau_1) + \alpha_1 \eta_1,
\]
\[
\frac{dv}{dt} = \alpha_{21} y' u(t - \tau_2) + \alpha_{22} y' v(t - \tau_2) + \alpha_2 \eta_2. \] (47)

where, $\tilde{A}_1 = - \frac{\alpha (1 - m)x'}{(ay + (1 - m)x)'x}$ and $A_1 = - \frac{r}{K}$.

Taking Fourier transform of both sides of each of the equation in system (47), we obtain
\[
\alpha_1 \tilde{\eta}_1(\omega) = (i\omega + \tilde{A}_1 x') \tilde{u}(\omega) + \alpha_{12} x' \tilde{v}(\omega) + A_1 x' \tilde{u}(\omega) e^{-i\omega \tau_1},
\]
\[
\alpha_2 \tilde{\eta}_2(\omega) = \alpha_{21} y' \tilde{u}(\omega) e^{-i\omega \tau_2} + (i\omega + \alpha_{22} y') \tilde{v}(\omega) e^{-i\omega \tau_2}. \] (48)

where $\tilde{u}(\omega) = \int_{-\infty}^{\infty} u(t) e^{-i\omega t} dt$.

The system of Eq. (48) can be written in matrix form as
\[
AX = B,
\] (49)
where,
\[
A = \begin{pmatrix} \tilde{A}_1 + A_1 e^{-i\omega \tau_1} + i\omega & \alpha_{12} \\ \alpha_{21} e^{-i\omega \tau_2} + i\omega & \alpha_{22} e^{-i\omega \tau_2} + i\omega \end{pmatrix}, \quad X = \begin{pmatrix} \tilde{u}(\omega) \\ \tilde{v}(\omega) \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} \alpha_1 \tilde{\eta}_1 \\ \alpha_2 \tilde{\eta}_2 \end{pmatrix}.
Now
\[ M = \det A = \{ (\tilde{a}_1 \alpha_{22} - \alpha_{12} \alpha_{21}) \cos (\omega \tau_2) + A_1 \alpha_{22} \cos (\omega (\tau_1 + \tau_2)) + \omega A_1 \sin (\omega \tau_1) + \omega \alpha_{22} \sin (\omega \tau_2) - A_1 \alpha_{22} \sin (\omega (\tau_1 + \tau_2)) + \omega \alpha_{22} \cos (\omega \tau_2) + \omega \alpha_1 \} \]

Assuming \( A^{-1} \) exists, we have \( A^{-1} = (a_{ij})_{2 \times 2} \), where
\[
\begin{align*}
a_{11} &= \frac{\alpha_{22} \cos (\omega \tau_2) + i \{ \omega - \alpha_{22} \sin (\omega \tau_2) \}}{M}, \\
a_{21} &= \frac{-a_2 \{ \cos (\omega \tau_2) + i \sin (\omega \tau_2) \}}{M}, \\
a_{12} &= \frac{-\alpha_{12}}{M}, \\
a_{22} &= \frac{\tilde{a}_1 + A_1 \cos (\omega \tau_1) + i \{ \omega - A_1 \sin (\omega \tau_1) \}}{M}.
\end{align*}
\]

Then the solution of system (49) can be written as
\[
\tilde{u}(s) = \sum_{j=1}^{2} a_{1j} \tilde{\eta}_j, \quad \tilde{v}(s) = \sum_{j=1}^{2} a_{2j} \tilde{\eta}_j 
\tag{50}
\]

Now following [51,52] and using Eq. (34), the spectral density of \( u \) is given by
\[
S_u(\omega) = \lim_{T \to \infty} \frac{1}{T^2} \int_{-T/2}^{T/2} \int_{-T/2}^{T/2} \{ u(t) u(t') \} \exp(i\omega(t' - t)) dt dt' = \sum_{j=1}^{2} |a_{1j}|^2 S_{\eta_j}(\omega).
\]

Similarly the spectral density of \( v \) is given by
\[
S_v = \sum_{j=1}^{2} |a_{2j}|^2 S_{\eta_j}(\omega).
\]

Therefore the fluctuation intensity (variance) of \( u \) is given by
\[
\sigma_u^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_u(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{j=1}^{2} |a_{1j}|^2 S_{\eta_j}(\omega) d\omega = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{j=1}^{2} |a_{1j}|^2 d\omega,
\]

since \( S_{\eta_j}(\omega) = 1 \).

Similarly the fluctuation intensity of \( v \) is given by
\[
\sigma_v^2 = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \sum_{j=1}^{2} |a_{2j}|^2 d\omega.
\]

6. Numerical computations

We perform numerical computations to observe various dynamics of the coexistence equilibrium for the model system (1). We consider the fixed parameter values as \( r = 3, K = 100, \alpha = 0.7, \gamma = 10, s = 3.5, h = 2, k = 30, m = 0.35 \) and initial value is considered as \( (x_0, y_0) = (12, 5) \) for each simulation. With these parameter values \( x^* = 98.6652, y^* = 47.0662 \). First we observe the two dimensional bifurcation scenario of the system (1) in \( \tau_1 \tau_2 \)-plane (Fig. 1). Red line is the Hopf-bifurcation line, i.e., at any coordinate \((\tau_1, \tau_2)\) on this red line, system experiences Hopf-bifurcation and at these points system switches its stability. Below and above regions of this line are respectively the stable and unstable regions of the system. If we choose any coordinate \((\tau_1, \tau_2)\) from below the red line, i.e., from the stable region, then all the solutions of the system (1) converges to the coexistence equilibrium point \( E^*(x^*, y^*) \) and if we choose any coordinate \((\tau_1, \tau_2)\) from above the red line, i.e., from the unstable region, then all the solutions of the system (1) oscillates around the coexistence equilibrium point \( E^*(x^*, y^*) \).

These two different characteristic dynamical phenomena will be presented numerically in this section. We shall show five different numerical examples corresponding to five analytical results [Case (I)–Case (V)].

(i) For Case (I): First, we consider the non-delayed system of model system (1). Here \( A + C_1 + C_2 = 6.4552 > 0 \) and \( D_2 + E = 10.3430 > 0 \). In this case, the system is asymptotically stable (time series in Fig. 2(i) and phase-plane in Fig. 2(ii)). Now we study the dynamical behavior of the delay-induced model system (1).

(ii) For Case (II): When \( \tau_1 \neq 0, \tau_2 = 0 \), one can compute from Theorem 2.4, that \( \tilde{a}_0 = 2.9600, \tau_{10} = 0.5301 \) and \( (A + C)^2 - C_1 - 2D_2 = 3.4887 > 0 \) and \( D_2^2 - E^2 = -107.3262 < 0 \). Therefore, the coexisting equilibrium \( E^*(x^*, y^*) \) is asymptotically stable for \( \tau_1 = 0.45 < \tilde{\tau}_{10} = 0.5301 \) (see Fig. 3 (ii)) and unstable for \( \tau_1 = 0.55 > \tilde{\tau}_{10} = 0.5301 \) (see Fig. 3 (iii)). When \( \tau_1 = \tilde{\tau}_{10} \), the model system (1) undergoes a Hopf-bifurcation at \( E^*(x^*, y^*) \). System behaviors in Case II can be demonstrated more prominently if we plot the bifurcation diagram in the three-dimensional space \( (\tau_1, x, y) \). Fig. 3 (i) shows that, when \( \tau_2 = 0 \), the coexistence equilibrium is stable for \( \tau_1 < \tilde{\tau}_{10} = 0.5301 \) but the instability sets in when \( \tau_1 > \tilde{\tau}_{10} = 0.5301 \).
Fig. 1. Stability regions of the system (1) are depicted in $\tau_1, \tau_2$-plane where the system shows stable and unstable dynamics in ‘Stable Region’ and ‘Unstable Region’ respectively and stable and unstable regions are separated by the red line which is denoted by ‘Hopf bifurcation line’ on which the system switches its stability through Hopf-bifurcation. Parameters are as in the text. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article).

Fig. 2. Time series (Fig. (i)) and phase-plane (Fig. (ii)) of the non-delayed ($\tau_1 = \tau_2 = 0$) counterpart of the system (1). This figure shows that the coexistence equilibrium is asymptotically stable. Parameters are as in the text.
Example 1. Consider the following model system:

\[
\begin{align*}
\frac{dx}{dt} &= 3x \left(1 - \frac{x(t - \tau_1)}{120} - \frac{0.7(1 - 0.25)xy}{12y + (1 - 0.25)x}\right), \\
\frac{dy}{dt} &= 3.7y \left[1 - \frac{2y(t - \tau_2)}{30 + (1 - 0.25)x(t - \tau_2)}\right].
\end{align*}
\]

This system has coexistence equilibrium point as \(E^*(x^* = 118.4439, y^* = 59.4165)\) and the analytical results presented in Theorems (2.3)–(2.7) has been verified. The numerical results for the existence of Hopf-bifurcation behavior around \(E^*\) has been presented for each cases with different values of delay parameters \(\tau_1\) and \(\tau_2\) in Table 1.
Fig. 4. Total dynamical feature of the system (1) with respect to \( \tau_2 \) in Case III (\( \tau_1 = 0.45 \in [0, 0.5301] \)). (i) Bifurcation diagram of the system (1) with respect to \( \tau_2 \) when \( \tau_1 = 0.45 \) in the three-dimensional space \((\tau_2, x, y)\). (ii) and (iii) are the time evolutions for \( \tau_2 = 0.4 < \bar{\tau}_2 = 0.4444 \) (system is stable) and \( \tau_2 = 0.48 > \bar{\tau}_2 = 0.4488 \) (system is unstable) respectively. A Hopf-bifurcation exists at \( \tau_2 = 0.4488 \). Other parameters are same as given in the text.

Fig. 5. (i) Bifurcation diagrams of the system (1) with respect to \( \tau_2 \) in Case IV (\( \tau_1 = 0 \)) in the three-dimensional space \((\tau_2, x, y)\). (ii) Time series for \((\tau_2 < \bar{\tau}_2 = 0.4235)\) and (iii) Time series for \((\tau_2 > \bar{\tau}_2 = 0.4235)\). These figures show that the coexistence equilibrium is stable for \( \tau_2 = 0.4 < 0.4235 \), unstable for \( \tau_2 = 0.44 > 0.4235 \) and a Hopf bifurcation exists at \( \tau_2 = 0.4235 \). Other parameters are same as given in the text.
Fig. 6. Bifurcation diagram of the system (1) with respect to $\tau_1$ in Case V ($\tau_2 = 0.4 \in [0, 0.4235]$) is depicted by (i) in the three-dimensional space ($\tau_1, x, y$). (ii) and (iii) are respectively the time evolutions for $\tau_1 = 0.48 < \tau_{1b} = 0.5278$ (system is stable) and $\tau_1 = 0.54 > \tau_{1b} = 0.5278$ (system is unstable). A Hopf-bifurcation exists at $\tau_1 = 0.5278$. Other parameters are same as given in the text.

Fig. 7. Figs. (i) and (ii) depict, respectively, the time evolution and corresponding phase trajectory of the stochastic model system (25) with very low intensity. Figures indicate that the system is stochastically stable. Parameters are given in the text.
Fig. 8. Figs. (i) and (ii) depict, respectively, the time evolution and corresponding phase trajectory of the stochastic model system (25) with intermediate intensity. Figures indicate that the system is in stochastic oscillatory mode. Parameters are given in the text.

Fig. 9. Figs. (i) and (ii) depict, respectively, the time evolution and corresponding phase trajectory of the stochastic model system (25) with very high intensity. Figures indicate that the system is stochastically chaotic. Parameters are given in the text.
Fig. 10. Plots (i, iii) and (ii, iv) depict, respectively, the time evolution and corresponding phase trajectory of the delay-stochastic model system (46). For $\tau_1 = 0.48$, $\tau_2 = 0.4$, system is stochastically stable (Figs. (i, ii)) and for $\tau_1 = 0.54$, $\tau_2 = 0.4$, system is stochastically unstable (Figs. (iii, iv)). Other parameters are given in the text.

Case I: $\tau_1 = \tau_2 = 0$

Case II(i): $\tau_1 = 0.48$, $\tau_2 = 0$

Case II(ii): $\tau_1 = 0.56$, $\tau_2 = 0$

Case III(i): $\tau_1 = 0.48$, $\tau_2 = 0.38$

Case III(ii): $\tau_1 = 0.48$, $\tau_2 = 0.47$

Case IV(i): $\tau_1 = 0$, $\tau_2 = 0.31$

Case IV(ii): $\tau_1 = 0$, $\tau_2 = 0.45$

Case V(i): $\tau_1 = 0.47$, $\tau_2 = 0.28$

Case V(ii): $\tau_1 = 0.54$, $\tau_2 = 0.28$

Fig. 11. Time evolution of Example 1. For case I, when $\tau_1 = \tau_2 = 0$, the positive interior equilibrium $E^* (118.4439, 59.4165)$ is locally asymptotically stable. For case II ($\tau_2 = 0$), system remains stable at $\tau_1 = 0.48 < \tilde{\tau}_1 = 0.5299$ and it shows stable periodic solution at $\tau_1 = 0.56 > \tilde{\tau}_1 = 0.5299$. Similar dynamics are observed for other cases (III–V).
Fig. 12. Time evolution of Example 2. For case I, when $\tau_1 = \tau_2 = 0$, the positive interior equilibrium $E^*(0.0573, 38.4007)$ is locally asymptotically stable. For case II ($\tau_2 = 0$), system remains stable at $\tau_1 = 0.5 < \tau_{1c} = 0.5658$ and it shows stable periodic solution at $\tau_1 = 0.6 > \tau_{1c} = 0.5658$. Similar dynamics are observed for other cases.

### Table 1
Numerical behavior of Example 1.

<table>
<thead>
<tr>
<th>Case</th>
<th>Condition</th>
<th>Critical value</th>
<th>Delay value</th>
<th>Status</th>
<th>Fig. 11</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$A + C_1 + C_2 = 6.6568 &gt; 0$, $D_2 + E = 10.9520 &gt; 0$</td>
<td>-</td>
<td>-</td>
<td>Stable</td>
<td>Case I</td>
</tr>
<tr>
<td>II</td>
<td>$(A + C_1)^2 - C_1^2 - 2D_2 = 4.8981 &gt; 0$, $D_2^2 - E^2 = -120.0352 &lt; 0$</td>
<td>$\bar{\omega}_0 = 2.9627$, $\tau_1 = 0.48$</td>
<td>Unstable</td>
<td>Case II(i)</td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>$\tau_1 = 0.48 \in [0, 0.5299]$</td>
<td>$\bar{\omega}_1 = 3.235$, $\tau_2 = 0.38$</td>
<td>Stable</td>
<td>Case III(i)</td>
<td></td>
</tr>
<tr>
<td>IV</td>
<td>$(A + C_1)^2 - C_1^2 - 2D_2 = 4.9474 &lt; 0$, $-D_2^2 + E^2 = -119.9470 &lt; 0$</td>
<td>$\omega_0 = 4.3263$, $\tau_2 = 0.45$</td>
<td>Unstable</td>
<td>Case IV(ii)</td>
<td></td>
</tr>
<tr>
<td>V</td>
<td>$\tau_2 = 0.28 \in [0, 0.3629]$</td>
<td>$\omega_1 = 3.002$, $\tau_1 = 0.54$</td>
<td>Unstable</td>
<td>Case V(ii)</td>
<td></td>
</tr>
</tbody>
</table>

### Table 2
Numerical behavior of Example 2.

<table>
<thead>
<tr>
<th>Case</th>
<th>Condition</th>
<th>Critical value</th>
<th>Delay value</th>
<th>Status</th>
<th>Fig. 12</th>
</tr>
</thead>
<tbody>
<tr>
<td>I</td>
<td>$A + C_1 + C_2 = 6.4700 &gt; 0$, $D_2 + E = 10.2576 &gt; 0$</td>
<td>-</td>
<td>-</td>
<td>Stable</td>
<td>Case I</td>
</tr>
<tr>
<td>II</td>
<td>$(A + C_1)^2 - C_1^2 - 2D_2 = 5.9800 &gt; 0$, $D_2^2 - E^2 = -105.3156 &lt; 0$</td>
<td>$\bar{\omega}_0 = 2.7747$, $\tau_1 = 0.5$</td>
<td>Stable</td>
<td>Case II(i)</td>
<td></td>
</tr>
<tr>
<td>III</td>
<td>$\tau_1 = 0.5 \in [0, 0.5658]$</td>
<td>$\bar{\omega}_1 = 3.107$, $\tau_2 = 0.39$</td>
<td>Unstable</td>
<td>Case III(i)</td>
<td></td>
</tr>
<tr>
<td>IV</td>
<td>$(A + C_1)^2 - C_1^2 - 2D_2 = 6.0170 &lt; 0$, $-D_2^2 + E^2 = -105.2187 &lt; 0$</td>
<td>$\omega_0 = 4.8602$, $\tau_2 = 0.38$</td>
<td>Unstable</td>
<td>Case IV(ii)</td>
<td></td>
</tr>
<tr>
<td>V</td>
<td>$\tau_2 = 0.29 \in [0, 0.3231]$</td>
<td>$\omega_1 = 3.32$, $\tau_1 = 0.56$</td>
<td>Unstable</td>
<td>Case V(ii)</td>
<td></td>
</tr>
</tbody>
</table>
Example 2. Consider the following model system with a new set of parameters:

\[
\frac{dx}{dt} = 2.8x \left[ 1 - \frac{x(t - \tau_1)}{100} \right] - \frac{0.5(1 - 0.45)xy}{9y + (1 - 0.45)x}.
\]
\[
\frac{dy}{dt} = 3.7y \left[ 1 - \frac{2.2y(t - \tau_2)}{30 + (1 - 0.45)x(t - \tau_2)} \right].
\]

This system has coexistence equilibrium point as \(x^* = 99.0573, y^* = 38.4007\) and the analytical results are verified. The numerical results are presented in Table 2 for different values of delay parameters \(\tau_1\) and \(\tau_2\). Time evolution of these examples (Examples 1 and 2) are presented in Figs. 11 and 12.

7. Conclusion and discussion

In this work, a prey–predator model that incorporates different biological delays and the effect of prey refuge is studied. A time delay \(\tau_1\) is considered in the logistic prey growth rate to represent density dependent feedback mechanism and the second time delay \(\tau_2\) is considered in the predator’s growth function to represent its gestation delay. We have studied the dynamic behavior of a multi delayed predator–prey system in the presence of prey refuge. Effect of prey refuge has an extended impact on the predator–prey models. However, both field and laboratory experiments confirm that intensity of prey refuge reduces predation rates by decreasing encounter rates between predator and prey. On the other hand, a predator–prey model becomes more realistic in the presence of different delays which are unavoidable elements in physiological and ecological processes.

It is also observed that the non-delayed system is asymptotically stable under some parametric restrictions. System experiences Hopf-bifurcation when the delay parameter attains a particular critical value. The interplay between two delays for fixed value of prey refuge has also been determined. It is noticed that these delays work in a complementary fashion. In other words, to keep the system in stable condition, the delay in the logistic prey growth should be low when gestation delay is high or vice versa. Thus, a two dimensional predator–prey system may exhibit simple stable behavior, regular cyclic behavior depending on the length of delays.

The main purpose of this work is to observe the extent to which positive and negative feedback delays along with prey refuge drive the population dynamics of a prey–predator interaction under fluctuating environment. Results show that prey refuge plays a significant role to change the stochastic stability of the system. To study the effect of environmental fluctuation on the non-delayed prey–predator system, we have superimposed Gaussian white noises and then studied non-equilibrium fluctuation and stability of the resulting stochastic model by using Fourier transform technique. Also the deterministic system and the noise-induced stochastic system may behave alike with respect to stability.

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